GRASSMANNIANS OF ABSOLUTE BANACH–COLMEZ SPACES

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ABSTRACT. We introduce Grassmannians of absolute Banach–Colmez space and prove the k-Grassmannian of a space of slope d/n is perfectoid if and only if $d/n < 1/k$. For $d = 1$, this proves a conjecture of Hansen–Mann. The key ingredient is a Plücker morphism, which realizes the k-Grassmannian of a slope d/n space as a Zariski closed subspace of the projectivization of a slope kd/n space. We also give a construction of invariant elements for profinite group actions on perfectoid Tate–Huber pairs, and use this to prove spaces of slope < 1 have perfectoid projectivizations. As applications, we establish vanishing bounds for Scholze's mod p Jacquet–Langlands functor on parabolic inductions, and also prove its vanishing in top degree on every infinite-dimensional non-supersingular irreducible representation.

CONTENTS

INTRODUCTION

Introduced by Colmez in [Col02], Banach–Colmez (BC) spaces play a key role in p-adic Hodge theory. Fargues and Fontaine [FF18] gave them a geometric interpretation through the cohomology of vector bundles on the Fargues–Fontaine curve. Le Bras [LB18] further developed this interpretation categorically, viewing BC spaces as v-sheaves of $\mathbb{Q}_p$-vector spaces on Perf_C and relating them to coherent sheaves on the curve.

Allowing the base to vary without choosing an untilt gives absolute BC spaces [Far20], which are v-sheaves on $\mathrm{Perf}_{\mathbb{F}_q}$. Their projectivizations include familiar moduli spaces: for $d \geq 1$, the quotient of the nonzero sections of $\mathcal{O}(d)$ by scalar multiplication is isomorphic to the space Div^d of effective Cartier divisors of degree d [Far20]. Motivated by the spaces $W_{n,k}$ of Hansen–Mann [HM22], we introduce Grassmannians of absolute BC spaces of positive slope and higher rank, extending the projectivizations studied in [Far20][FS24]. The primary goal of this article is to study geometric properties of these Grassmannians, and our main result determines when are they representable by perfectoid spaces.

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Let $K/\mathbb{Q}_p$ be a finite extension with uniformizer π a dn residue field $\mathbb{F}_q$. Let C be a complete algebraically closed extension of K. Write $\mathbf{B}^{\varphi^n=\pi^d}$ for the absolute BC space of rank n and degree $d>0$; its slope is given by $\mu = d/n$. As a v-sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$, $\mathbf{B}^{\varphi^n=\pi^d}$ is defined by

$$\mathbf{B}^{\varphi^n=\pi^d} : S \mapsto H^0(X_S, \mathcal{E}_{n,d}), \quad S \in \mathrm{Perf}_{\mathbb{F}_q}$$

where $\mathcal{E}_{n,d}$ is the standard semistable vector bundlef of rank n and degree d. For $0 < k < n$, let $\mathcal{I}\mathrm{nj}(\mathcal{O}^k, \mathcal{E}_{n,d})$ be the v-sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ parametrizing stably injective homomorphisms $\mathcal{O}^k \hookrightarrow \mathcal{E}_{n,d}$. We define the k-Grassmannian of $\mathbf{B}^{\varphi^n=\pi^d}$ as

$$\mathrm{Gr}_k(\mathbf{B}^{\varphi^n=\pi^d}) := \mathcal{I}\mathrm{nj}(\mathcal{O}^k, \mathcal{E}_{n,d})/\underline{\mathrm{GL}_k(K)}$$

and write $\mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d})$ its base change to $\mathrm{Spd}\, C$. Our main result is the following:

Theorem A (Theorem 4.14 and Proposition 5.6)**.** *The k-Grassmannian $\mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d})$ is perfectoid if and only if $\mathbf{B}^{\varphi^n=\pi^d}$ has slope $\mu < 1/k$.*

For $d = 1$, the Grassmannian $\mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi})$ is isomorphic to the space $W_{n,n-k}$ defined in [HM22, Definition 5.1]. Theorem A therefore proves [HM22, Conjecture 5.4].

Let us now explain the key ideas and constructions leading to Theorem A. Classically, the Plücker embedding provides a link between different Grassmannians; the same also holds in our setting, except now an additional parameter, the degree d, is involved. Given a space $\mathbf{B}^{\varphi^n=\pi^d}$ of positive slope, put $N = \binom{n}{k}$ and $D = d\binom{n-1}{k-1}$. These are the rank and degree of $\bigwedge^k \mathcal{E}_{n,d}$. Taking the determinant defines a Plücker morphism

$$\mathrm{Pl} : \mathrm{Gr}_k(\mathbf{B}^{\varphi^n=\pi^d}) \to \mathrm{Gr}_1(\mathbf{B}^{\varphi^N=\pi^D}),$$

where a stably injective morphism $i : \mathcal{F} \hookrightarrow \mathcal{E}_{n,d}$ with $\mathcal{F} \cong \mathcal{O}^k$ at every geometric point is sent to $\wedge^k i : \det \mathcal{F} \hookrightarrow \bigwedge^k \mathcal{E}_{n,d}$. We then show:

Theorem B (Theorem 4.11 and Proposition 4.13)**.** *If $\mathbf{B}^{\varphi^n=\pi^d}$ has slope $< 1/k$, the Plücker morphism $\mathrm{Pl} : \mathrm{Gr}_k(\mathbf{B}^{\varphi^n=\pi^d}) \to \mathrm{Gr}_1(\mathbf{B}^{\varphi^N=\pi^D})$ is a Zariski closed immersion.*

Since $D/N = kd/n$, Theorem B (after base change to $\mathrm{Spd}\, C$) reduces perfectoidness of the k-Grassmannian of a space of slope $< 1/k$ to the case of a projectivized space of slope < 1. In this range, the absolute BC space is identified with the special fiber of the universal cover of a p-divisible group. By [SW13, Proposition 3.1.3], its punctured space, after quotienting by the torsion subgroup $\mu_{q-1} \subset \mathcal{O}_K^\times$, admits a finite $\mathcal{O}_K^\times$-stable cover by affinoid perfectoid $\mathcal{V}_i = \mathrm{Spa}(V_i, V_i^+)$, where

$$V_i = \mathbb{F}_q(\!(u_i^{1/p^\infty})\!)\langle y_{ij}^{1/p^\infty}, 0 \le j \ne i \le d-1\rangle$$
$$V_i^+ = \mathbb{F}_q[\![u_i^{1/p^\infty}]\!]\langle y_{ij}^{1/p^\infty}, 0 \le j \ne i \le d-1\rangle$$

For each unit subgroup $U_m = 1 + \pi^m \mathcal{O}_K \subset \mathcal{O}_K^\times$, we construct (Definition 2.9) certain norms for elements in V_i with respect to U_m by averaging its action and taking the limit. In particular, the resulting U_m-norm of u_i is a pseudouniformizer of $V_i^{U_m}$, and the invariant pair $(V_i^{U_m}, V_i^{+U_m})$ is perfectoid. To identify (V_i, V_i) as a pro-étale U_1-torsor over $(V_i^{U_1}, V_i^{+U_1})$, the essential step is to show the quotient at level U_{m+1} is a finite étale $\mathbb{F}_q$-torsor over the one at level U_m. For this, we construct (Lemma 3.3) an element $\theta_m \in V_i^{U_{m+1}}$ such that $g(\theta_m) - \theta_m \in (V_i^{U_{m+1}})^\times$ for every nontrivial $g \in \mathbb{F}_q$. We also show (Proposition 3.2) such a condition in fact gives a general criterion for finite étale torsors. The constructions above yield the following result:

Theorem C (Theorem 3.7). *If $\mathbf{B}^{\varphi^n=\pi^d}$ has slope <1, the v-sheaf quotients*

$$(\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\})/\underline{\mathcal{O}_K^\times} \quad \textit{and} \quad (\mathbf{B}_C^{\varphi^n=\pi^d}\setminus\{0\})/\underline{K^\times}$$

are representable by perfectoid spaces.

Together, Theorems B and C prove the positive direction of the main theorem. For the converse, suppose $d/n \geq 1/k$, one observes that the Grassmannian then contains a locus of injections which factor through the subbundle $\mathcal{O}(1/k) \subset \mathcal{E}_{n,d}$. This locus parametrizes exactly the degree 1 modifications of a vector bundle fibrewise isomorphic to $\mathcal{O}^k$, and we show it is Zariski closed in the Grassmannian. For $k=1$, this locus is simply Div_C^1, which is not perfectoid (Proposition 5.1). For $k>1$, consider the sublocus of modifications supported on the canonical divisor $i : S \to X_{S^\flat}$. This is the $\mathrm{GL}_k(K)$-quotient of the infinite-level Lubin–Tate space $\mathscr{M}_\infty^{\mathrm{LT}}$, hence isomorphic to $\mathbb{P}^{k-1}$. As $\mathbb{P}^{k-1}$ is not perfectoid, this completes the other direction of the main theorem.

Theorem A has applications to Scholze's mod p Jacquet–Langlands functor $\mathcal{J}_n^i$. Scholze's general bound [Sch18, Theorem 1.1] gives $\mathcal{J}_n^i(\rho)=0$ for $i>2(n-1)$ and every admissible $\mathbb{F}_p$-representation ρ of $\mathrm{GL}_n(K)$. For $1 \leq k \leq n-1$, denote by $P_k \subset \mathrm{GL}_n$ the standard maximal parabolic subgroup of block size $(n-k,k)$. Hansen–Mann's partial Künneth formula [HM22, Theorem 5.2] expresses Scholze's functor on parabolic inductions from $P_k(K)$ interms of cohomology of $\mathrm{Gr}_{n-k,C}(\mathbf{B}^{\varphi^n=\pi})$, with coefficients in a complex of étale $\mathbb{F}_p$-sheaves involving $\mathcal{J}_k$. Since this Grassmannian has dimension $n-k$ and the étale cohomology of a perfectoid space vanishes in degrees larger than its dimension, we obtain the following vanishing bounds for all admissible proper parabolic inductions for $\mathrm{GL}_n(K)$.

Theorem D (Theorem 6.5). *Let $0<k<n$. For any $\rho \in \mathrm{Rep}_{\mathbb{F}_p}^{\mathrm{adm}}(P_k(K))$, one has*

$$\mathcal{J}_n^i(\mathrm{Ind}_{P_k(K)}^{\mathrm{GL}_n(K)}(\rho)) = 0 \quad \textit{for } i>n+k-2.$$

Theorem D further gives a vanishing theorem for non-supersingular admssible representations in the top degree $2(n-1)$. Scholze's general bound implies that $\mathcal{J}_n^{2n-2}$ is right-exact. The classification theorem of Henniart–Vignéras [HV19] shows that every non-supersingular irreducible $\rho \in \mathrm{Rep}_{\mathbb{F}_p}^{\mathrm{adm}}(\mathrm{GL}_n(K))$ is isomorphic to a parabolic induction of some Steinberg type representation. Upon further restricting to such ρ which are infinite-dimensional, one gets a surjection from a proper parabolic induction to ρ, from which we deduce the following vanishing result.

Theorem E (Theorem 6.9). *For any $n>1$ and any irreducible $\rho \in \mathrm{Rep}_{\mathbb{F}_p}^{\mathrm{adm}}(\mathrm{GL}_n(K))$ that is non-supergingular and infinite-dimensional, we have $\mathcal{J}^{2n-2}(\rho)=0$.*

Therefore, for infinite-dimensional irreducible admissible representations of $\mathrm{GL}_n(K)$, Scholze's bound can be improved to $2n-3$. Via explicit examples, one can see this new bound is in fact sharp, and the assumption of infinite-dimension is also crucial.

Outline of the article. For $S \in \mathrm{Perf}_{\mathbb{F}_q}$, let $X_S = X_{S,K}$ be the relative Fargues–Fontaine curve. Fix an absolute BC space $\mathbf{B}^{\varphi^n=\pi^d}$ of slope $\mu>0$.

§1 reviews vector bundles on X_S, defines the *Grassmannians* $\mathrm{Gr}_k(\mathbf{B}^{\varphi^n=\pi^d})$ for $0<k<n$, and surveys known results on their representability.

§2 introduces the notion of *orbit norm* for elements of an $\mathbb{F}_p$-algebra endowed with an action of a filtered profinite group. After covering the punctured BC space by affinoid perfectoid charts, it proves their rings of $\mathcal{O}_K^\times$-invariants are perfectoid when $\mu < 1$.

§3 realizes this geometrically by showing each chart is a pro-étale $\mathcal{O}_K^\times$-torsor over the adic spectrum of its invariant ring. For $\mathbf{B}^{\varphi^n=\pi^d}$ with $\mu < 1$, tt then proves its projectivization is perfectoid after base change to $\operatorname{Spd} C$.

§4 constructs the *Plücker morphisms* and identifies the k-Grassmannian of $\mathbf{B}^{\varphi^n=\pi^d}$ with the decomposable locus in the projectivization of a BC space of slope $k\mu$. It deduces that the k-Grassmannian is perfectoid when $\mu < 1/k$.

§5 treats the rest cases of $\mu \geq 1/k$. It proves that Div_C^1 is not perfectoid, and then establishes the non-perfectoidness of the k-Grassmannian by reduction to Div_C^1 or P^{k-1}.

§6 presents applications to representation theory. It recalls Scholze's functor, proves vanishing bounds of the functor on parabolic inductions and non-supersingular representations, and gives examples with explicit computations.

Notation and conventions. We let $K/\mathbb{Q}_p$ be a finite extension, with uniformizer π and residue field $\mathbb{F}_q$. We use K_n to denote the unramified degree n extension of K. For an $\mathbb{F}_p$-algebra R, $W(R)$ denotes its ring of Witt vectors and $[\cdot] : R \to W(R)$ denotes the Teichmuller lift. The (relative) Fargues–Fontaine curve will be defined with respect to K. For $m \in N^*$, $D_{1/m}$ is the division algebra over K of invariant $1/m$.

We fix F a complete algebraically closed nonarchimedean field over $\mathbb{F}_q$, $C/\mathbb{Q}_p$ a complete algebraically closed nonarchimedean field with ring of integers $\mathcal{O}_C$. The category of perfectoid spaces over $\mathbb{F}_q$ (resp. over C) is denoted $\operatorname{Perf}_{\mathbb{F}_q}$ (resp. Perf_C).

We reserve the letters n, d for the rank and degree of an absolute Banach–Colmez space $\mathbf{B}^{\varphi^n=\pi^d}$ and always assume $n \in \mathbb{N}^*, d \in \mathbb{Z}$.

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1. Absolute Banach–Colmez spaces

In this section, we first recall vectors bundles on the relative Fargues–Fontaine curve and its relationship to absolute BC spaces following [Far20] and [FS24, Chapter II]. We also review the moduli spaces $X_{n,k}$ and $W_{n,k}$ from [JLH21, Appendix A] and [HM22, Section 5], which parametrize homomorphisms between vector bundles, and compare them with the punctured absolute BC spaces. We then introduce Grassmannians of absolute BC spaces, and survey some previous results of their representability.

1.1. Vector bundles on the Fargues–Fontaine curve and absolute BC spaces.

For $S = \operatorname{Spa}(R, R^+) \in \operatorname{Perf}_{\mathbb{F}_q}$ with pseudo-uniformizer $\varpi \in R^+$, denote by

$$Y_S = Y_{S,K} := \operatorname{Spa}(W_{\mathcal{O}_K}(R^+)) \backslash V(\pi[\varpi]).$$

The *relative Fargues–Fontaine curve* is the quotient $X_S = X_{S,K} := Y_S/\varphi^{\mathbb{Z}}$. Via gluing, such constructions also extend to general $S \in \mathrm{Perf}_{\mathbb{F}_q}$. For an interval $I = [a, b] \subset (0, \infty)$ with rational ends, denote by

$$Y_{S,I} := \{|\pi|^b \leq |[\varpi]| \leq |\pi|^a\} \subset Y_S,$$

which is an open subspace of Y_S. In this notation, Y_S can be also written as $Y_{S,(0,\infty)}$.

We write $\mathbf{B}$ for v-sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ given by $\mathbf{B}(S) = \mathcal{O}(Y_S)$. It has a Frobenius endomorphism $\varphi = \varphi_q$ coming from that of Y_S. When $S = \mathrm{Spa}(R, R^+)$ is affinoid, we also denote $\mathcal{O}(Y_S)$ by B_S and $\mathcal{O}(Y_{S,I})$ by $B_{S,I}$. The open subspace $Y_{S,I}$ is then given by $\mathrm{Spa}(B_{S,I}, B^+_{S,I})$, hence affinoid.

Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$ and assume $d \in \mathbb{Z}$, $n \in \mathbb{N}^*$. When n, d are relatively prime, we write $\mathcal{O}(d/n)$ for the stable vector bundle of slope d/n on X_S. For a generic pair (d, n), we write $\mathcal{E}_{n,d}$ for the semistable vector bundle of degree d and rank n on X_S. According to the classification of vector bundles on $X_F = X_{\mathrm{Spa}(F,F^+)}$ [FF18], if $g = \gcd(d, n)$, we have

$$\mathcal{E}_{n,d} = \mathcal{O}\left(\frac{d/g}{n/g}\right)^{\oplus g}.$$

Moreover, any vector bundle $\mathcal{E}/X_F$ of slope μ is always isomorphic to a direct sum of vector bundles of them form $\mathcal{O}_{X_F}(d/n)$, and $H^0(X_F, \mathcal{E}) \neq 0$ if and only if $\mu \geq 0$.

For a vector bundle $\mathcal{E}$ on X_S, the *Banach–Colmez space* $\mathrm{BC}(\mathcal{E})$ associated to $\mathcal{E}$ is the locally spatial diamond over Perf_S defined by

$$\mathrm{BC}(\mathcal{E})(T) = H^0(X_T, \mathcal{E}_T), \quad T \in \mathrm{Perf}_S\,.$$

For $d \geq 0$, the *absolute Banach-Colmez space* $\mathbf{B}^{\varphi^n=\pi^d}$ is the v-sheaf over $\mathrm{Perf}_{\mathbb{F}_q}$ defined by

$$\mathbf{B}^{\varphi^n=\pi^d} = \mathcal{O}(Y_S)^{\varphi^n=\pi^d} = H^0(X_S, \mathcal{E}_{n,d}), \quad S \in \mathrm{Perf}_{\mathbb{F}_q}\,.$$

Notice that $\mathbf{B}^{\varphi^n=\pi^d}$ is not a diamond. For example, when $d = 0$ and $n = 1$, one gets $\mathbf{B}^{\varphi^n=\mathrm{id}} = \underline{K}$. However, pullback to $S \in \mathrm{Perf}_{\mathbb{F}_q}$ gives

$$\mathbf{B}_S^{\varphi^n=\pi^d} := \mathbf{B}^{\varphi^n=\pi^d} \times_{\mathrm{Spd}\,\mathbb{F}_q} S = \mathrm{BC}(\mathcal{E}_{n,d}),$$

hence $\mathbf{B}_S^{\varphi^n=\pi^d}$ is a locally spatial diamond over S. This also explains the notion "absolute" (see [Far20, §7] for more details).

Proposition 1.1 ([FS24, Proposition II.2.5 (iv)]). *For $0 < d \leq n \cdot [K : \mathbb{Q}_p]$, there is an isomorphism of v-sheaves*

$$\mathbf{B}^{\varphi^n=\pi^d} \cong \mathrm{Spd}\,\mathbb{F}_q[\![x_0^{1/p^\infty}, \ldots, x_{d-1}^{1/p^\infty}]\!].$$

Proof. The statement in [FS24] is originally over $\overline{\mathbb{F}}_q$; its proof uses [SW13, Proposition 3.1.3 (iii)], which holds for any perfect ring of characteristic p. $\square$

Notice the presentation above only involves the degree d of $\mathbf{B}^{\varphi^n=\pi^d}$. As is later explained in Lemma 2.1, the rank n is related to the Frobenius action and also the π-action on the variables x_i. Also from this presentation, we see again $\mathbf{B}^{\varphi^n=\pi^d}$ is not a diamond as it contains the non-analytic point $\mathrm{Spd}\,\mathbb{F}_q$.

Let $\mathbf{B}^{\varphi^n=\pi^d}\backslash\{0\}$ be the v-subsheaf of $\mathbf{B}^{\varphi^n=\pi^d}$ containing sections which are nonzero after pullback to any geometric point, which is called a *punctured absolute Banach–Colmez* *space*. Specifically, when $0 < d \leq n \cdot [K : \mathbb{Q}_p]$, we have

$$\begin{aligned}\mathbf{B}^{\varphi^n = \pi^d} \setminus \{0\} &\cong \operatorname{Spd}\left(\mathbb{F}_q[\![x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]\!]\right) \setminus V(x_0, \dots, x_{d-1}) \\ &= \bigcup_{i=0}^{d-1} D(x_i)\end{aligned}$$

where each $D(x_i)$ is an open perfectoid ball over the perfectoid field $\mathbb{F}_q(\!(x_i^{1/p^\infty})\!)$. Therefore, we have:

Proposition 1.2. *For $0 < d \leq n \cdot [K : \mathbb{Q}_p]$, the v-sheaf $\mathbf{B}^{\varphi^n = \pi^d} \setminus \{0\}$ is representable by a perfectoid space.*

1.2. **Homomorphisms between vector bundles.** We now recall certain functors on $\mathrm{Perf}_{\mathbb{F}_q}$ parametrizing homomorphisms between vector bundles following [BFHHL+22] and [JLH21, Appendix A]. We then compare them with the punctured absolute BC spaces, and define Grassmannians of absolute BC spaces.

Let $\mathcal{E}, \mathcal{F}$ be vector bundles over X_F. The Hom-functor $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ sends $S \in \mathrm{Perf}_{\mathbb{F}_q}$ to the set of $\mathcal{O}_{X_S}$-module homomorphisms $\mathcal{E}_{X_S} \to \mathcal{F}_{X_S}$ and we have $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})(S) \cong H^0(X_S, \mathcal{E}^\vee \otimes \mathcal{F})$, where $\mathcal{E}^\vee$ denotes the dual vector bundle of $\mathcal{E}$. Denote by $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F}) \subseteq \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ the subfunctor parametrizing surjective homomorphisms. Also denote by $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F}) \subseteq \mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$ the subfunctor parametrizing *stably injective* homomorphisms, i.e. maps $\mathcal{E}_{X_S} \to \mathcal{F}_{X_S}$ which are injective after pullback along any geometric point $\mathrm{Spa}(F, F^+) \to S$. Both $\mathcal{I}\mathrm{nj}(\mathcal{E}, \mathcal{F})$ and $\mathcal{S}\mathrm{urj}(\mathcal{E}, \mathcal{F})$ are open subfunctors of $\mathcal{H}\mathrm{om}(\mathcal{E}, \mathcal{F})$, and all three functors are locally spatial diamonds over $\operatorname{Spd} \mathbb{F}_q$ (see [BFHHL+22, §3] and [Sch26, Proposition 13.4]).

Directly from their definitions, one notices that

$$\mathcal{I}\mathrm{nj}(\mathcal{O}, \mathcal{E}_{n,d}) \cong \mathbf{B}^{\varphi^n = \pi^d} \setminus \{0\}.$$

Moreover, $\underline{K^\times}$ acts on both sides and the isomorphism between their S-points is $K^\times$-equivariant. Thus we have an isomorphism of quotient sheaves

$$\mathcal{I}\mathrm{nj}(\mathcal{O}, \mathcal{E}_{n,d}) / \underline{K^\times} \cong (\mathbf{B}^{\varphi^n = \pi^d} \setminus \{0\}) / \underline{K^\times}, \tag{1.1}$$

where the right hand side is a projectivized absolute BC space. The idea of studying such projectivized absolute BC spaces first appeared in [Far20]. Specifically, for $d \geq 0$, it was shown in loc. cit. that there is an isomorphism

$$(\mathbf{B}^{\varphi = \pi^d} \setminus \{0\}) / \underline{K}^\times \cong \mathrm{Div}^d$$

where Div^d is the sheaf sending $S \in \mathrm{Perf}_{\mathbb{F}_q}$ to the set of closed degree-d Cartier divisors of X_S. However, such an identification with the moduli space of Cartier divisors does not exist for $\mathcal{I}\mathrm{nj}(\mathcal{O}, \mathcal{E}_{n,d}) / \underline{K}^\times$ when $n > 1$, in which case one instead has

$$(\mathcal{I}\mathrm{nj}(\mathcal{O}, \mathcal{E}_{n,d})) \times_{\operatorname{Spd} \mathbb{F}_q} \operatorname{Spd} \mathbb{F}_{q^n}) / \underline{K_n^\times} \cong \mathrm{Div}^d \times_{\operatorname{Spd} \mathbb{F}_q} \operatorname{Spd} \mathbb{F}_{q^n}.$$

Nevertheless, the quotient $\mathcal{I}\mathrm{nj}(\mathcal{O}, \mathcal{E}_{n,d}) / \underline{K^\times}$ for $n > 1$ naturally leads to some other moduli spaces. As one can see from (1.1), $\mathcal{I}\mathrm{nj}(\mathcal{O}, \mathcal{E}_{n,d}) / \underline{K}^\times$ parametrizes lines inside $\mathbf{B}^{\varphi^n = \pi^d}$. Generalizing on this idea, we may also parametrize higher dimensional subspaces of $\mathbf{B}^{\varphi^n = \pi^d}$ by considering stably injective morphisms from $\mathcal{O}^k$ to $\mathcal{E}_{n,d}$.

Definition 1.3. Let $n, d \in \mathbb{N}^*$ and $1 \leq k \leq n-1$. We define the *k-Grassmannian* of the absolute Banach–Colmez space $\mathbf{B}^{\varphi^n=\pi^d}$ as the quotient sheaf

$$\mathrm{Gr}_k(\mathbf{B}^{\varphi^n=\pi^d}) := \mathcal{I}\mathrm{nj}(\mathcal{O}^k, \mathcal{E}_{n,d})/\underline{\mathrm{GL}_k(K)}$$

and we denote by $\mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d})$ its base change $\mathrm{Gr}_k(\mathbf{B}^{\varphi^n=\pi^d}) \times_{\mathrm{Spd}\,\mathbb{F}_q} \mathrm{Spd}\, C$.

1.3. **Representability of Grassmannians.** After base change to $\mathrm{Spd}\, C$, the Grassmannian $\mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d})$ is a locally spatial diamond. The main goal of this text is to study when are such Grassmannians representable by perfectoid spaces. The first examples of perfectoid Grassmannians come from the auxiliary spaces $X_{n,k}$ and $W_{n,k}$ studied in [JLH21] and [HM22], and Definition 1.3 is directly motivated by such spaces.

We now recall the construction of $X_{n,k}$ and $W_{n,k}$. Let $\mathscr{M}_\infty^{\mathrm{LT}}$ be the infinite-level Lubin–Tate space over C, which is perfectoid and admits a continuous $\mathrm{GL}_n(K) \times D_{1/n}^\times$-action [SW13]. The quotient $\mathscr{M}_\infty^{\mathrm{LT}}/\,\mathrm{GL}_n(K)$ is identified with the projective space $\mathbb{P}^{n-1}$. For $1 \leq k \leq n-1$, let $P_k(K) \subseteq \mathrm{GL}_n(K)$ be the maximal parabolic subgroup of the form

$$P_k(K) = \begin{pmatrix} \mathrm{GL}_{n-k}(K) & * \\ 0 & \mathrm{GL}_k(K) \end{pmatrix}.$$

The space $X_{n,k}$ is defined as the quotient $\mathscr{M}_\infty^{\mathrm{LT}}/P_k(K)$. In the meantime, by passing to its associated diamond, $X_{n,k}$ also admits a moduli interpretation in terms of vector bundles on the relative Fargues–Fontaine curve. By [SW13], the diamond $\mathscr{M}_\infty^{\mathrm{LT},\diamond}$ associated to $\mathscr{M}_\infty^{\mathrm{LT}}$ can be identified with the functor which assigns to $S \in \mathrm{Perf}_C$ the set of modifications

$$0 \to \mathcal{O}^n \to \mathcal{O}(1/n) \to i_*W \to 0 \tag{1.2}$$

over $X_{S^\flat}$, where $i : S \to X_{S^\flat}$ is the canonical closed immersion and W is some rank one projective $\mathcal{O}_S$-module. By [JLH21, Theorem A.1.1], the quotient sheaf $\mathscr{M}_\infty^{\mathrm{LT},\diamond}/\underline{P_k(K)}$ then assigns to $S \in \mathrm{Perf}_C$ the set of diagrams

$$\mathcal{O}(1/n) \twoheadrightarrow \mathcal{E} \hookleftarrow \mathcal{F} \tag{1.3}$$

of vector bundles over $X_{S^\flat}$ with $\mathcal{E} \cong \mathcal{O}(1/k)$ and $\mathcal{F} \cong \mathcal{O}^k$ at all geometric points of $S^\flat$ and $\mathcal{F} \hookrightarrow \mathcal{E}$ a modification of $\mathcal{F}$ as in (1.2). The space $W_{n,k}$ is then defined as the functor on Perf_C which parametrizes surjections of the form $\mathcal{O}(1/n) \twoheadrightarrow \mathcal{E}$ over $X_{S^\flat}$ with $\mathcal{E} \cong \mathcal{O}(1/k)$ at all geometric points of $S^\flat$. In other words, $W_{n,k}$ remembers only the left half of (1.3).

Before stating the representability results of $X_{n,k}$ and $W_{n,k}$, we first describe their relationship with the Grassmannians defined in Definition 1.3. Since $\mathcal{O}(1/n)$ is stable, taking the kernel of a surjection $\mathcal{O}(1/n) \twoheadrightarrow \mathcal{E}$ as in (1.3) gives a stably injective morphism $\mathcal{E}' \hookrightarrow \mathcal{O}(1/n)$ with $\mathcal{E}' \cong \mathcal{O}^{n-k}$ at every geometric point. Thus we have[1]

$$\begin{aligned} W_{n,k} &= (\mathcal{S}\mathrm{urj}(\mathcal{O}(1/n), \mathcal{O}(1/k)) \times_{\mathrm{Spd}\,\mathbb{F}_q} \mathrm{Spd}\, C)/\underline{D_{1/k}^\times} \\ &\cong (\mathcal{I}\mathrm{nj}(\mathcal{O}^{n-k}, \mathcal{O}(1/n)) \times_{\mathrm{Spd}\,\mathbb{F}_q} \mathrm{Spd}\, C)/\underline{\mathrm{GL}_{n-k}(K)} && (1.4) \\ &= \mathrm{Gr}_{n-k,C}(\mathbf{B}^{\varphi^n=\pi}). && (1.5) \end{aligned}$$

The space $X_{n,k}$ is not perfectoid when $k > 1$ ([JLH21, Theorem A.1.5]); when $k = 1$, one sees from (1.3) that $X_{n,1} = W_{n,1}$, hence we have

$$X_{n,1} \cong \mathrm{Gr}_{n-1,C}(\mathbf{B}^{\varphi^n=\pi}). \tag{1.6}$$

[1] In [JLH21, Appendix A], Hanse defined the $\mathcal{I}\mathrm{nj}$-functor and the $\mathcal{S}\mathrm{urj}$-functor directly over Perf_C, whereas we defined them over $\mathbb{F}_q$; hence we need to base change to $\mathrm{Spd}\, C$ to recover the space $W_{n,k}$.

Theorem 1.4 ([JLH21, Theorem 4.2.1]). *The diamond $X_{n,1}$ is representable by a perfectoid space.*

Johansson and Ludwig proved this result by first exhibiting $\mathscr{M}_\infty^{\mathrm{LT}}/P_1(K)$ as an open subspace of some infinite level Harris–Taylor Shimura variety. They then proved the latter is perfectoid via global methods.

Theorem 1.5 ([JLH21, Corollary A.1.4]). *The diamond $W_{n,n-1}$ is representable by a perfectoid space.*

Hansen's proof was originally written for the case of $n = 2$, which also applies to all $n > 1$. We briefly recall the argument here.

Proof. As in (1.4), we have $W_{n,n-1} \cong (\mathcal{I}\mathrm{nj}(\mathcal{O}, \mathcal{O}(\frac{1}{n})) \times_{\mathrm{Spd}\,\mathbb{F}_q} \mathrm{Spd}\, C)/\underline{K^\times}$. Let E/K be the unramified degree n extension and let E_∞/E be the Lubin–Tate extension of E, which has Galois group $\mathrm{Gal}(E_\infty/E) \cong \mathcal{O}_E^\times$. Then for $S \in \mathrm{Perf}_{\mathbb{F}_{q^n}}$, we have

$$\mathcal{I}\mathrm{nj}(\mathcal{O}, \mathcal{O}(1/n))(S) = H^0(X_{S,K}, \mathcal{O}(1/n))^\times = H^0(X_{S,K_n}, \mathcal{O}(1))^\times, \tag{1.7}$$

and [Wei17, Proposition 3.5.3] further gives an $E^\times$-equivariant isomorphism

$$H^0(X_{S,K_n}, \mathcal{O}(1))^\times \cong \mathrm{Spd}\, \hat{E}_\infty^\flat. \tag{1.8}$$

Therefore, combining (1.7) and (1.8), we get

$$\mathcal{I}\mathrm{nj}(\mathcal{O}, \mathcal{O}(1/n)) \times_{\mathrm{Spd}\,\mathbb{F}_q} \mathrm{Spd}\,\mathbb{F}_{q^n} \cong \mathrm{Spd}\, \hat{E}_\infty^\flat.$$

Since $\hat{E}$ is perfectoid and $\mathrm{Gal}(E_\infty/E) \cong \mathcal{O}_E^\times$, the subfield $L := E_\infty^{\mathcal{O}_K^\times}$ is again deeply ramified over K, hence $\hat{L}$ is perfectoid. Thus we get

$$\begin{aligned}
W_{n,n-1} &\cong (\mathrm{Spd}\, \hat{E}_\infty^\flat \times_{\mathrm{Spd}_{\mathbb{F}_{q^n}}} \mathrm{Spd}\, C)/\underline{K^\times} \\
&\cong (\mathrm{Spd}\, \hat{E}_\infty^\flat/\underline{\mathcal{O}_K^\times} \times_{\mathrm{Spd}\,\mathbb{F}_{q^n}} \mathrm{Spd}\, C)/(\varphi_n \times \mathrm{id})^{\mathbb{Z}} \\
&\cong (\mathrm{Spd}\, \hat{L}^\flat \times_{\mathrm{Spd}\,\mathbb{F}_{q^n}} \mathrm{Spd}\, C)/(\varphi_n \times \mathrm{id})^{\mathbb{Z}},
\end{aligned}$$

where φ_n is the Frobenius of K_n. Since both $\hat{L}^\flat$ and C are perfectoid and $\varphi_n \times \mathrm{id}$ acts properly discontinuously on the product, $W_{n,n-1}$ is perfectoid. $\square$

These two theorems above give the perfectoidness of $W_{n,k}$ at the two ends when $k = 1$ and $k = n-1$. Hansen and Mann then conjectured the following:

Conjecture 1.6 ([HM22, Conjecture 5.4]). *For ever $1 \le k \le n-1$, $W_{n,k}$ is perfectoid.*

In terms of Grassmannians, Theorem 1.4 and Theorem 1.5 show that $\mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi})$ is perfectoid when $k = 1$ and $k = n-1$; the conjecture above further predicted the perfectoidness of all other Grassmannians (over $\mathrm{Spd}\, C$) of the slope $1/n$ absolute BC space $\mathbf{B}^{\varphi^n=\pi^d}$.

2. Orbit norm of formal coordinates

In this section, we describe the $\mathcal{O}_K^\times$-action on an absolute Banach-Colmez space $\mathbf{B}^{\varphi^n=\pi^d}$, and fix an $\mathcal{O}_K^\times$-stable affinoid perfectoid open cover $\bigcup \mathcal{W}_i$ of the punctured space $\mathbf{B}^{\varphi^n=\pi^d} \setminus \{0\}$. For each $\mathcal{W}_i = \mathrm{Spa}(W_i, W_i^+)$, we construct $\mathcal{O}_K^\times$-invariant elements in W_i, and use them to show each quotient Tate–Huber pair $(W_i^{\mathcal{O}_K^\times}, W_i^{+\mathcal{O}_K^\times})$ is perfectoid. Unless otherwise specified, we assume $0 < d < n$ throughout this section.

2.1. **Punctured BC spaces and $\mathcal{O}_K^\times$-action.** Recall from Proposition 1.2 that when $0 < d \leq n \cdot [K : \mathbb{Q}_p]$, the punctured absolute BC space $\mathbf{B}^{\varphi^n=\pi^d} \backslash \{0\}$ is perfectoid and admits a presentation

$$\mathbf{B}^{\varphi^n=\pi^d} \backslash \{0\} \cong \operatorname{Spd} \mathbb{F}_q[\![x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]\!] \backslash V(x_0, \dots, x_{d-1}).$$

The formal variables x_i arise from the coordinates of a formal $\mathcal{O}_K$-module and parametrize topoligically nilpotent elements of rings, thus we have $0 \leq |x_i| < 1$ for all i. Since the variables do not vanish simultaneously, $\mathbf{B}^{\varphi^n=\pi^d} \backslash \{0\}$ admits an open cover

$$\mathbf{B}^{\varphi^n=\pi^d} \backslash \{0\} \cong \bigcup_{i=0}^{d-1} \mathcal{W}_i, \quad \mathcal{W}_i = \{|x_i| \neq 0, |x_j| \leq |x_i| \text{ for every } j\}. \tag{2.1}$$

Let $y_{ij} = x_j/x_i$ for $j \neq i$, we then have $0 \leq |y_{ij}| \leq 1$ over W_i. Since x_i is invertible over W_i, we can write $\mathcal{W}_i$ more explicitely as $\operatorname{Spa}(W_i, W_i^+)$ where

$$W_i := \mathbb{F}_q(\!(x_i^{1/p^\infty})\!)\langle y_{ij}^{1/p^\infty}, j \neq i\rangle \quad \text{and} \quad W_i^+ := \mathbb{F}_q[\![x_i^{1/p^\infty}]\!]\langle y_{ij}^{1/p^\infty}, j \neq i\rangle.$$

Therefore, $\bigcup_{i=0}^{d-1} \mathcal{W}_i$ gives an affinoid perfectoid open cover of $\mathbf{B}^{\varphi^n=\pi^d} \backslash \{0\}$.

The absolute BC space $\mathbf{B}^{\varphi^n=\pi^d}$ is a K-vector space object in the category of v-sheaves on $\operatorname{Perf}_{\mathbb{F}_q}$. Thus it admits an action of $K^\times$. We now describe and fix the $K^\times$-action on the formal variables that we use for later computations.

Lemma 2.1. *The formal variables $x_0, ..., x_{d-1}$ of $\mathbf{B}^{\varphi^n=\pi^d}$ can be chosen such that*

(i) $\pi \in \mathcal{O}_K^\times$ acts on x_i by

$$\pi(x_i) = \begin{cases} x_{i-1}^q & \text{if } 1 \leq i \leq d-1 \\ x_{d-1}^{q^{n-d+1}} & \text{if } i = 0 \end{cases}$$

(ii) $c \in \mathbb{F}_q^\times$ with Teichmüller lift $[c] \in \mathcal{O}_K^\times$ acts on x_i by

$$[c](x_i) = cx_i$$

for all $0 \leq i \leq d-1$.

Proof. Let us first relabel the formal variables of $\mathbf{B}^{\varphi^n=\pi^d}$ as $\xi_0, \dots, \xi_i$. For affinoid perfectoid $S = \operatorname{Spa}(R, R^+) \in \operatorname{Perf}_{\mathbb{F}_q}$, an S-point of $\operatorname{Spd} \mathbb{F}_q[\![\xi_0^{1/p^\infty}, \dots, \xi_{d-1}^{1/p^\infty}]\!]$ is a tuple $(r_0, \dots, r_{d-1}) \in (R^{\circ\circ})^d$ where r_i is the value of the coordinate ξ_i. By [FF18, Proposition 4.2.1], there is a bijection

$$\begin{aligned} \mathscr{L}_{d,n} : (R^{\circ\circ})^d &\xrightarrow{\sim} \mathbf{B}^{\varphi^n=\pi^d}(R) \\ (r_0, ..., r_{d-1}) &\mapsto \sum_{i=0}^{d-1} \sum_{m \in \mathbb{Z}} [r_i^{q^{-mn}}] \pi^{md+i}. \end{aligned}$$

Multiplying the image by π, we get $\pi \mathscr{L}_{d,n}(r_0, \dots, r_{d-1}) = \mathscr{L}_{d,n}(r_{d-1}^{q^n}, r_0, \dots, r_{d-2})$. Matching this with the left hand side of the map $\mathscr{L}_{d,n}$, we see that π acts on the coordinates (ξ_i) by

$$\pi(\xi_0) = \xi_{d-1}^{q^n}, \quad \pi(\xi_i) = \xi_{i-1} \text{ for } 1 \leq i \leq d-1.$$

For each i, let $x_i = \xi_i^{q^i}$. This is an invertible change of coordinates because R is a perfect ring. The π-action on the variables (x_i) is then given by

$$\pi(x_0) = \xi_{d-1}^{q^n} = x_{d-1}^{q^{n-d+1}}, \quad \pi(x_i) = \xi_{i-1}^{q^i} = x_{i-1}^q \text{ for } 1 \leq i \leq d-1.$$

For $c \in \mathbb{F}_q^\times$, since $c^q = c$, multiplication by its Teichmüller lift satisfies

$$[c]\mathscr{L}_{d,n}(\xi_0, \dots, \xi_{d-1}) = \mathscr{L}_{d,n}(c\xi_0, \dots, c\xi_{d-1}).$$

Hence we get $[c](x_i) = (c\xi_i)^{q^i} = cx_i$ as desired. $\square$

We fix these formal variables (x_i) for $\mathbf{B}^{\varphi^n=\pi^d}$ together with the $K^\times$-action described in this lemma. Notice that our choice of (x_i) is analogous to the ones used in [BHHMS25, § 2.3]; their construction is the special case when $K/\mathbb{Q}_p$ is unramified.

The formal variables (x_i) can be identified with the coordinates of a formal $\mathcal{O}_K^\times$-module $\mathcal{G}_{d,n}$ of dimension d and height n over $\mathbb{F}_q$ (see [FF18, Exemple 4.3.2]). Let $\widetilde{\mathcal{G}}_{d,n} = \varprojlim_{[\pi]} \mathcal{G}_{d,n}$ be its universal cover. By [FF18, Proposition 4.4.5], we have an isomorphism $\widetilde{\mathcal{G}}_{d,n} \cong \mathbf{B}^{\varphi^n=\pi^d}$. We let $F = (F_0, \dots, F_{d-1})$ denote the formal group law of $\mathcal{G}_{d,n}$ in the chosen coordinates. For the d-dimensional formal variables X, Y of $\mathcal{G}_{d,n}$, since $F(X, 0) = X$ and $F(0, Y) = Y$, the i-th component of $F(X, Y)$ is given by $F_i(X, Y) = X_i + Y_i + C_i(X, Y)$, where $C_i(X, Y) \in (X_0, \dots, X_{d-1})(Y_0, \dots, Y_{d-1})$ are the mixed terms.

Lemma 2.2. *Let $I \subseteq \mathbb{F}_q[\![x_0, \dots, x_{d-1}]\!]$ be the ideal generated by $x_0, \dots, x_{d-1}$.*

(i) For every $a \in \mathcal{O}_K^\times$ with residue $\overline{a} \in \mathbb{F}_q^\times$, we have

$$a(x_i) \equiv \overline{a}x_i \mod I^q.$$

(ii) For $a = 1 + \pi b$ with $b \in \mathcal{O}_K$, we have

$$a(x_i) \equiv x_i + \overline{b}\pi(x_i) \mod I^{q+1}.$$

(iii) For $a = 1 + \pi^m b$ with $m \geq 1$ and $b \in \mathcal{O}_K$, write $m = cd + f$ with $c \geq 0$ and $0 \leq f < d$. We then have

$$a(x_i) \equiv x_i \mod I^{q^{cn+f}}.$$

Proof. For $a \in \mathcal{O}_K^\times$, write $a = [\overline{a}] + \pi a_1$ with $a_1 \in \mathcal{O}_K$. By Lemma 2.1, we have $[\overline{a}](x_i) = \overline{a}x_i$. As every scalar endomorphism fixes the origin of $\widetilde{\mathcal{G}}_{d,n}$, we have $a_1(I) \subseteq I$. Lemma 2.1 also implies $\pi(I) \subseteq I^q$, from which we get $(\pi a_1)(I) \subseteq I^q$. Let $X = (x_0, \dots, x_{d-1})$. By the formal group law, we have $a(x_i) = F_i([\overline{a}](X), (\pi a_1)(X))$. Since the second factor lies in I^q, every mixed term lies in I^{q+1}, hence $a(x_i) \equiv \overline{a}x_i \mod I^q$. This proves part (i), and part (ii) is the special case of (i) when $\overline{a} = 1$.

For part (iii), we first compute the action of π^m. Notice that $\pi^{cd}(x_i) = x_i^{q^{cn}}$; in the remaining f iterations of π-action, the cyclic index crosses the special case $x_0 \mapsto x_{d-1}^{q^{n-d+1}}$ of part (i) of Lemma 2.1 exactly when $i < f$. Thus we have

$$\pi^m(x_i) = \begin{cases} x_{i-f}^{q^{cn+f}} & \text{if } i \geq f \\ x_{i-f+d}^{q^{cn+f+n-d}} & \text{if } i < f. \end{cases}$$

Because $d < n$ by assumption, the exponent in both cases is at least q^{cn+f}, hence we have $\pi^m I \subseteq I^{q^{cn+f}}$. By part (i), we have $b(x_i) \equiv \overline{b}x_i \mod I^q$, which implies $\pi^m b(x_i) \equiv \overline{b}\pi^m(x_i) \mod I^{q^{cn+f+1}}$ and $\pi^m b(I) \subseteq I^{q^{cn+f}}$. Since $a(x_i) = F_i(X, \pi^m b(X))$, the mixed terms belong to $I \cdot I^{q^{cn+f}} = I^{1+q^{cn+f}}$. As $q^{cn+f+1} > 1 + q^{cn+f}$, we get $a(x_i) \equiv x_i + \overline{b}\pi^m(x_i) \mod I^{1+q^{cn+f}}$. Since $\pi^m(x_i) \in I^{q^{cn+f}}$, this proves part (iii). $\square$

Proposition 2.3. *Each affinoid perfectoid $\mathcal{W}_i$ in the open cover (2.1) of $\mathbf{B}^{\varphi^n=\pi^d} \setminus \{0\}$ is stable under the action of $\mathcal{O}_K^\times$.*

Proof. Recall that each $\mathcal{W}_i$ is defined by the condition $|x_j| \leq |x_i| \neq 0$ for all j. For $a \in \mathcal{O}_K^\times$, we show $|a(x_j)| \leq |a(x_i)| \neq 0$ for all j. By part (i) of Lemma 2.2, we have $a(x_j) = \bar{a}x_j + \epsilon_j$ for some $\epsilon_j \in I^q$. Since the variables x_j are topologically nilpotent, we have $0 < |x_j| < 1$. Since $\epsilon_j \in I^q$, every monomial in ϵ_j has total degree at least q, hence we get $|\epsilon_j| \leq |x_i|^q < |x_i|$. Because $|\bar{a}| = 1$, we also have $|\epsilon_i| < |x_i| = |\bar{a}x_i|$. The non-archimedean triangle inequality then gives

$$|a(x_i)| = |\bar{a}x_i + \epsilon_i| = |x_i|\,, \quad |a(x_j)| \leq \max\{|x_j|\,, |x_i|^q\} \leq |x_i|\,.$$

Therefore each $\mathcal{W}_i$ is $\mathcal{O}_K^\times$-stable. □

2.2. **Tame quotients.** The group $\mathcal{O}_K^\times$ has a decomposition

$$\mathcal{O}_K^\times \cong \mu_{q-1} \times U_1,$$

where $\mu_{q-1} = [\mathbb{F}_q^\times] \subseteq \mathcal{O}_K^\times$ is the *tame subgroup* and $U_1 = 1 + \pi\mathcal{O}_K$ is the *principal unit group*. For $m \geq 1$, let $U_m = 1 + \pi^m\mathcal{O}_K$ be the *m-th higher unit group* of $\mathcal{O}_K^\times$. Such U_m form a descending filtration

$$\mathcal{O}_K^\times \supseteq U_1 \supseteq U_2 \supseteq \cdots$$

with quotients $\mathcal{O}_K^\times/U_1 \cong \mathbb{F}_q^\times$ and $U_m/U_{m+1} \cong \mathbb{F}_q$. We now describe the tame quotient $\mathcal{W}_i/\underline{\mu_{q-1}}$ for each chart $\mathcal{W}_i$ of the punctured space.

Proposition 2.4. *Let $0 \leq i \leq d-1$ and let $u_i = x_i^{q-1}$.*

(i) For each i, let (V_i, V_i^+) be the perfectoid Tate–Huber pair given by

$$V_i = \mathbb{F}_q(\!(u_i^{1/p^\infty})\!)\langle y_{ij}^{1/p^\infty}, j \neq i\rangle, \quad V_i^+ = \mathbb{F}_q[\![u_i^{1/p^\infty}]\!]\langle y_{ij}^{1/p^\infty}, j \neq i\rangle.$$

Then for the affinoid perfectoid $\mathcal{W}_i = \mathrm{Spa}(W_i, W_i^+) \subseteq \mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\}$, we have $(W_i^{\mu_{q-1}}, W_i^{+\mu_{q-1}}) = (V_i, V_i^+)$.

(ii) Let $\mathcal{V}_i = \mathrm{Spa}(V_i, V_i^+)$. Then we have

$$\mathcal{W}_i/\underline{\mu_{q-1}} \cong \mathcal{V}_i \quad \textit{and} \quad (\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\})/\underline{\mu_{q-1}} \cong \bigcup_{i=0}^{d-1} \mathcal{V}_i.$$

In particular, the quotient $(\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\})/\underline{\mu_{q-1}}$ is a perfectoid space.

Proof. Let $c \in \mathbb{F}_q^\times$ and let $[c] \in \mu_{q-1}$ be its Teichmüller lift. From Lemma 2.1, we get $[c](u_i) = u_i$ and $[c](y_{ij}) = y_{ij}$. For p-power roots of x_i, Lemma 2.1 also implies $[c](x_i^{1/p^r}) = c^{1/p^r}x_i^{1/p^r}$, where c^{1/p^r} is uniquely determined in the perfect field $\mathbb{F}_q$. Hence we have

$$[c](u_i^{1/p^r}) = u_i^{1/p^r}, \quad [c](y_{ij}^{1/p^r}) = y_{ij}^{1/p^r},$$

which shows that $V_i \subseteq (W_i)^{\mu_{q-1}}$ and $V_i^+ \subseteq (W_i^+)^{\mu_{q-1}}$. As $q-1$ is prime to p, the map $c \mapsto c^{1/p^r}$ from $\mathbb{F}_q^\times$ to $\mathbb{F}_q^\times$ is bijective. This gives $V_i = (W_i)^{\mu_{q-1}}$ and $V_i^+ = (W_i^+)^{\mu_{q-1}}$, which proves part (i).

Since the tame subgroup μ_{q-1} is finite, [CGJ23, Proposition 2.1.3] (cf. also [Han16, Theorem 1.2]) together with part (i) imply $\mathcal{W}_i/\underline{\mu_{q-1}} \cong \mathcal{V}_i$. Because the open cover $\bigcup_{i=0}^{d-1}\mathcal{W}_i$ of $\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\}$ is $\mathcal{O}_K^\times$-stable by Propoisition 2.3, such $\mathcal{V}_i$ form an affinoid perfectoid open cover of $(\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\})/\underline{\mu_{q-1}}$, which proves part (ii). □

From now on, we also fix the affinoid perfectoid cover

$$(\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\})/\underline{\mu_{q-1}} = \bigcup_{i=0}^{d-1}\mathcal{V}_i, \quad \mathcal{V}_i = \mathrm{Spa}(V_i, V_i^+)$$

as described in the proposition above.

On each chart $\mathcal{V}_i$, we have an action of U_m for every $m \geq 1$. Recall from Lemma 2.2 that the estimates of the $\mathcal{O}_K^\times$-action on the variables x_i are expressed in powers of the ideal $I = (x_0, \ldots, x_{d-1})$. On each chart $\mathcal{W}_i$ of $\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\}$, since x_i is a topologically nilpotent unit, we can describe such estimates in powers of x_i. Similarly on each chart $\mathcal{V}_i$ of the tame quotient, we can also describe such estimates in powers of u_i.

As one can see from part (iii) of Lemma 2.2, the x_i-adic order of the U_m-action is described in terms of m modulo the cycle length d. It is thus convenient to fix the following constant attached to any $m \in \mathbb{N}^*$.

Notation 2.5. For every $m \in \mathbb{N}^*$, write $m = cd + f$ with $0 \leq f \leq d-1$. We denote by

$$Q_m := \frac{q^{cn+f}-1}{q-1}.$$

For each higher unit subgroup $U_m \subset \mathcal{O}_K^\times$, we now give an estimate of its action on the elements $u_i, y_{ij} \in V_i^+$.

Lemma 2.6. *Let $m \geq 1$, $g \in U_m$, and $j \neq i$. We have*
- *(i) $g(u_i)/u_i - 1, g(y_{ij}) - y_{ij} \in u_i^{Q_m}V_i^+$;*
- *(ii) $g(u_i^\lambda V_i^+) = u_i^\lambda V_i^+$ for every $\lambda \in \mathbb{Z}[1/p]_{\geq 0}$.*

Proof. For any j, denote by $\delta_j := (g(x_j) - x_j)/x_i$. Part (iii) of Lemma 2.2 then gives $\delta_j \in x_i^{q^{cn+f}-1}W_i^+$. As g commutes with the action of μ_{q-1}, δ_j is μ_{q-1}-invariant, hence $\delta_j \in u_i^{Q_m}V_i^+$; in particular, we get $(1+\delta_j) \in (V_i^+)^\times$. Direct computation then gives

$$\frac{g(u_i)}{u_i} - 1 = (1+\delta_i)^{q-1} - 1, \quad g(y_{ij}) - y_{ij} = \frac{\delta_j - y_{ij}\delta_i}{1+\delta_i},$$

from which part (i) follows. Since $g(u_i)/u_i \in (V_i^+)^\times$, taking its p-power roots preserves the filtration induced by u_i. This proves part (ii). □

Lemma 2.7. *For every $m \geq 1$, the sequence $(Q_{m+r}/q^{r+1})_{r\geq 0}$ is increasing and tends to ∞ as $r \to \infty$.*

Proof. From the definition of Q_m, we have $Q_{m+1} \geq qQ_m$ and $Q_{m+d} \geq q^nQ_m$. The first inequality proves that the sequence is increasing, and the second one further gives

$$\frac{Q_{m+\ell d}}{q^{\ell d+1}} \geq q^{\ell(n-d)}\frac{Q_m}{q}$$

for any $\ell \geq 0$. Since $Q_m > 0$ and $d < n$, we get $Q_{m+r}/q^{r+1} \to \infty$ as $r \to \infty$. □

2.3. **Orbit norm.** Let G be a pro-p group equipped with a descending filtration

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

formed by open normal subgroups G_r such that $\bigcup_{r\geq 0} G_r = \{1\}$. For each quotient G/G_r, a *transversal* for G/G_r is a subset of G containing exactly one coset representative of each coset in G/G_r. For each $r \geq 1$, let Σ_r be a transversal for G/G_r. We call $(\Sigma_r)_{r\geq 1}$ a *compatible* system of transversals for (the filtration $G_\bullet$ of) G if there exist transversals Θ_r for G_{r-1}/G_r such that

$$\Sigma_r = \Theta_1 \cdots \Theta_r.$$

Let A be a complete perfect topological $\mathbb{F}_p$-algebra with an action of G as above.

Definition 2.8. Let $r \geq 0$ and let Σ_r be a transversal for G/G_r. For an element $x \in A$, we call

$$N_G^{(r)}(x) := \left(\prod_{\sigma \in \Sigma_r} \sigma(x)\right)^{1/[G:G_r]}$$

a *depth-r G-orbit norm* of x.

Notice that $N_G^{(r)}(x)$ depends on the choice of Σ_r. For fixed Σ_r, $N_G^{(r)}(x)$ is well-defined because A is perfect. Let $(\Sigma_r)_{r\geq 0}$ be a compatible system of transversals for G with associated $(N_G^{(r)}(x))_{r\geq 0}$. If $N_G^{(r)}(x)$ converges in A as $r \to \infty$ and the limit is independent of the chosen compatible transversals, we let

$$N_G(x) := \lim_{r\to\infty} N_G^{(r)}(x).$$

Definition 2.9. For $x \in A$, the element $N_G(x)$ defined above will be called the *G-orbit norm*[2] of x.

In the rest of this article, we shall be concerned only with the case when A is $V_i = \mathcal{O}(\mathcal{V}_i)$ or $W_i = \mathcal{O}(\mathcal{W}_i)$, and G is $U_m \subseteq \mathcal{O}_K^\times$ for some $m \geq 1$ with fixed filtration

$$G = U_m \supset G_1 = U_{m+1} \supset \cdots \supset G_r = U_{m+r} \supset \cdots$$

Examples 2.10.

(1) Suppose p is odd and $K = \mathbb{Q}_p$. Then we have $U_1 = 1 + p\mathbb{Z}_p$, which is topologically generated by $\gamma = 1 + p$. A compatible system of transversals for U_1 can be chosen as $\Sigma_r = \{\gamma^t : 0 \leq t < p^r\}$.

(2) For general finite extension $K/\mathbb{Q}_p$, a compatible system of transversals for U_1 can be chosen as follows. For $r \geq 1$, we choose transversals $\Theta_r \subseteq U_1$ for the quotients U_r/U_{r+1} as $\Theta_r = \{1 + \pi^r[c] : c \in \mathbb{F}_q\}$ and then set $\Sigma_r = \Theta_1 \cdots \Theta_r$. This gives $|\Sigma_r| = [U_1 : U_{1+r}] = q^r$, and Σ_r is explicitely given by

$$\Sigma_r = \{\prod_{t=1}^{r}(1 + \pi^t[c_t]) : c_1, \ldots, c_r \text{ vary independently over } \mathbb{F}_q\}.$$

The following technical lemma gives an estimate of the difference between orbit norms at successive depths.

Lemma 2.11. *Let $\lambda \in \mathbb{Z}[1/p]_{\geq 0}$, $f \in x_i^\lambda W_i^+$, and let $\{N_{U_m}^{(r)}(f)\}_{r\geq 0}$ be a compatible system of U_m-orbit norms of f. Suppose for every $r \geq 0$, there exists $e_r \in \mathbb{Z}[1/p]_{\geq 0}$ with $e_r \geq \lambda$ such that $g(f) - f \in x_i^{e_r} W_i^+$ for every $g \in U_{m+r}$, then we have*

$$N_{U_m}^{(r+1)}(f) - N_{U_m}^{(r)}(f) \in x_i^{\lambda + (e_r - \lambda)/q^{r+1}} W_i^+.$$

[2] The construction of $N_G^{(r)}(x)$ and $N_G(x)$ is motivated by an exercise given by Jared Weinstein at the Arizona Winter School in 2013; see [Wei13, Project A].

Proof. Let (Σ_r) be the compatible transversals inducing such orbit norms. Since they are compatible, there exists transversal Θ_{r+1} for U_{m+r}/U_{m+r+1} such that $\Sigma_{r+1} = \Sigma_r\Theta_{r+1}$. Set $P_r = \prod_{\sigma\in\Sigma_r}\sigma(f)$ so that $N^{(r)}_{U_m}(f)^{q^r} = P_r$. For $\sigma \in \Sigma_r$ and $\tau \in \Theta_{r+1}$, write

$$\sigma\tau(f) = \sigma(f) + \delta_{\sigma,\tau}, \quad \delta_{\sigma,\tau} = \sigma(\tau(f) - f).$$

Part (ii) of Lemma 2.6 implies

$$\sigma(f) \in x_i^{\lambda} W_i^+, \quad \delta_{\sigma,\tau} \in x_i^{e_r} W_i^+.$$

From the definition of $N^{(r)}_{U_m}(f)$, we have

$$P_{r+1} = \prod_{\sigma\in\Sigma_r}\prod_{\tau\in\Theta_{r+1}}(\sigma(f) + \delta_{\sigma,\tau}).$$

Within this product, terms containing no $\delta_{\sigma,\tau}$-factor have product $\prod_{\sigma\in\Sigma_r}\sigma(f)^q = P_r^q$. In the product of the rest terms, there are $k \geq 1$ factors of order at least e_r and $q^{r+1} - k$ factors of order at least λ, giving total order at least

$$ke_r + (q^{r+1} - k)\lambda = q^{r+1}\lambda + k(e_r - \lambda) \geq q^{r+1}\lambda + (e_r - \lambda).$$

Thus we have

$$P_{r+1} - P_r^q \in x_i^{q^{r+1}\lambda + (e_r-\lambda)} W_i^+.$$

Since W_i^+ is perfect of characteristic p, we have

$$(N^{(r+1)}_{U_m}(f) - N^{(r)}_{U_m}(f))^{q^{r+1}} = N^{(r+1)}_{U_m}(f)^{q^{r+1}} - N^{(r)}_{U_m}(f)^{q^{r+1}} = P_{r+1} - P_r^q.$$

Taking the unique q^{r+1}-th root then proves the claim. □

We now give a criterion for the existence of the orbit norm.

Lemma 2.12. *Under the hypothesis of Lemma 2.11, if we further have*

$$\frac{e_r - \lambda}{q^{r+1}} \to \infty,$$

then the orbit norm $N_{U_m}(f)$ exists and belongs to $W_i^{+U_m}$. Moreover, if $\beta \in \mathbb{Z}[1/p]_{\geq 0}$ satisfies $\lambda + (e_r - \lambda)/q^{r+1} \geq \beta$ for every $r \geq 0$, then $N_{U_m}(f) - f \in x_i^{\beta} W_i^+$.

Proof. By Lemma 2.11, we have

$$N^{(r+1)}_{U_m}(f) - N^{(r)}_{U_m}(f) \in x_i^{\lambda + (e_r-\lambda)/q^{r+1}} W_i^+. \tag{2.2}$$

As the exponent of x_i goes to infinity by assumption, the sequence $(N^{(r)}_{U_m}(f))_r$ is Cauchy and converges in W_i^+. We are left to show the limit is indepdent of the chosen transversals. Suppose $(\Sigma'_r)_r$ is another compatible system of transversals for $(U_m/U_{m+r})_r$ and let $N^{(r)}_{U_m,\Sigma'_r}(f)$ be the corresponding depth-r norms. For each Σ'_r, matching its representatives with those of Σ_r gives

$$\Sigma'_r = \{\sigma h_\sigma : \sigma \in \Sigma_r\}$$

where each h_σ belongs to U_{m+r}. By assumption of Lemma 2.11 and part (ii) of Lemma 2.6, we have $\sigma h_\sigma(f) - \sigma(f) \in x_i^{e_r} W_i^+$. Taking the product over the transversals Σ_r and Σ'_r respectively and taking the q^r-th root, we get

$$N^{(r)}_{U_m,\Sigma'_r}(f) - N^{(r)}_{U_m,\Sigma_r}(f) \in x_i^{\lambda + (e_r-\lambda)/q^r} W_i^+.$$

As the exponent of x_i goes to infinity, we see that $N^{(r)}_{U_m,\Sigma'_r}(f)$ converges to the same limit as $N^{(r)}_{U_m,\Sigma_r}(f)$. Thus the orbit norm $N_{U_m}(f)$ exists. To see that $N_{U_m}(f)$ is U_m-invariant, notice that any $g \in U_m$ gives another system of transversals $(g\Sigma_r)_r$. As the orbit norm is independent of the transversals, we get $g(N_{U_m}(f)) = N_{U_m}(f)$.

Finally, if the expotent in (2.2) is bounded below by β, summing up successive differences gives $N^{(r)}_{U_m}(f) - f \in x_i^{\beta} W_i^+$. Passing to the limit gives $N_{U_m}(f) - f \in x_i^{\beta} W_i^+$. $\square$

Remark 2.13. Notice that the U_m-orbit norm of f being U_m-invariant is almost formal and "built-in" from its construction. In general, given G a filtered pro-p group acting continuously on a complete perfect topological $\mathbb{F}_p$-algebra A, if $N_G(x)$ exists for some $x \in A$, it is natural to expect $N_G(x)$ is G-invariant.

For each chart $\mathcal{V}_i = \mathrm{Spa}(V_i, V_i^+)$ of the tame quotient $(\mathbf{B}^{\varphi^n=\pi^d} \setminus \{0\})/\underline{\mu_{q-1}}$, we now show that the elements $u_i, y_{ij} \in V_i^+ = \mathbb{F}_q[\![u_i^{1/p^\infty}]\!]\langle y_{ij}^{1/p^\infty}, j \neq i\rangle$ admit U_m-orbit norm for every higher unit group $U_m \subset \mathcal{O}_K^\times$.

Proposition 2.14. *Assume $0 < d < n$. For every chart $\mathcal{V}_i$ of $(\mathbf{B}^{\varphi^n=\pi^d} \setminus \{0\})/\underline{\mu_{q-1}}$ and every $m \geq 1$, the orbit norms $N_{U_m}(u_i)$ and $N_{U_m}(y_{ij})$ exist in V_i^+ and are U_m-invariant. Moreover, we have*

$$\frac{N_{U_m}(u_i)}{u_i} - 1 \in u_i^{Q_m/q} V_i^+, \quad N_{U_m}(y_{ij}) - y_{ij} \in u_i^{Q_m/q} V_i^+.$$

Proof. We first work in W_i^+. Let $r \geq 0$ and $g \in U_{m+r}$. Since $u_i = x_i^{q-1}$, we have $u_i \in x_i^{q-1} W_i^+$ and $y_{ij} \in W_i^+$, hence part (i) of Lemma 2.6 gives

$$\begin{aligned} g(u_i) - u_i &\in x_i^{(q-1)(Q_{m+r}+1)} W_i^+ \\ g(y_{ij}) - y_{ij} &\in x_i^{(q-1)Q_{m+r}} W_i^+. \end{aligned}$$

For both u_i and y_{ij}, the ratio $(e_r - \lambda)/q^{r+1}$ from Lemma 2.12 is given by

$$\frac{(q-1)(Q_{m+r}+1) - (q-1)}{q^{r+1}} = \frac{(q-1)Q_{m+r}}{q^{r+1}},$$

which goes to infinity as $r \to \infty$ by Lemma 2.7 and is bounded below by $(q-1)Q_m/q$. Applying Lemma 2.12 then shows the existence of $N_{U_m}(u_i)$ and $N_{U_m}(y_{ij})$, proves they are U_m-invariant, and also gives

$$N_{U_m}(u_i) - u_i \in x_i^{(q-1)(1+Q_m/q)} W_i^+, \quad N_{U_m}(y_{ij}) - y_{ij} \in x_i^{(q-1)Q_m/q} W_i^+.$$

Now passing to V_i^+, since V_i^+ is closed in W_i^+ and all the intermediate depth-r norms belong to V_i^+ by construction, we get $N_{U_m}(u_i), N_{U_m}(y_{ij}) \in V_i^{+U_m}$. $\square$

Corollary 2.15. *For every $m \geq 1$ and every $\ell \geq 0$, we have*

(i) $N_{U_m}(u_i)$ *is a topologically nilpotent unit in* V_i*;*
(ii) $N_{U_m}(u_i)^{1/p^\ell}, N_{U_m}(y_{ij})^{1/p^\ell}$ *belongs to* $V_i^{+U_m}$*;*
(iii) $N_{U_m}(u_i)^{1/p^\ell}$ *is a topologically nilpotent unit in* V_i*.*

Proof. Since $Q_m > 0$, Proposition 2.14 gives

$$\frac{N_{U_m}(u_i)}{u_i} \in 1 + u_i^{Q_m/q} V_i^+ \subset 1 + V_i^{\circ\circ} \subset (V_i^+)^\times.$$

Since u_i is a topologically nilpotent unit, we get (i) and (iii). Part (ii) follows directly from the perfectness of V_i^+. $\square$

2.4. **Perfectoid invariant rings.** We conclude this section by showing the $\mathcal{O}_K^\times$-invariant subring of each W_i is perfectoid.

Proposition 2.16. *For every $m \geq 1$, $(V_i^{U_m}, V_i^{+U_m})$ is a perfectoid Tate–Huber pair. In particular, $(W_i^{\mathcal{O}_K^\times}, W_i^{+\mathcal{O}_K^\times})$ is a perfectoid Tate–Huber pair.*

Proof. By [Sch26, Proposition 3.5], to show $V_i^{U_m}$ is perfectoid, it is enough to show it is a perfect complete Tate ring. By Proposition 2.14 and Corollary 2.15, $N_{U_m}(u_i)$ is a topologically nilpotent unit in $V_i^{+U_m}$ with $N_{U_m}(u_i) = u_i\epsilon_m$ for some $\epsilon_m \in (V_i^+)^\times$. Thus for every $s \geq 0$, we get $N_{U_m}(u_i)^s V^+ = u_i^s V^+$, where $\{u_i^s V_i^+\}_{s\geq 0}$ form a neighbourhood basis of 0 in V_i. Since we also have $V_i^{U_m} \cap N_{U_m}(u_i)^s V_i^+ = N_{U_m}(u_i)^s V_i^{+U_m}$, $\{N_{U_m}(u_i)^s V_i^{+U_m}\}_{s\geq 0}$ form a neighbourhood basis of 0 in $V_i^{U_m}$. Hence $V_i^{+U_m}$ is an open bounded ring of definition, and $V_i^{U_m}$ is a Tate ring with pseudouniformizer $N_{U_m}(u_i)$. As $V_i^{U_m} = \bigcap_{g\in U_m} \ker(g - \mathrm{id})$, $V_i^{U_m}$ is closed in V_i, hence complete. To show it is perfect, let $v \in V_i^{U_m}$ and let $w \in V_i$ be its unique p-th root. For every $g \in U_m$, we have $g(w)^p = g(v) = v = w^p$. Since V_i is perfect, this gives $g(w) = w$, hence $w \in V_i^{U_m}$. Thus Frobenius on $V_i^{U_m}$ is bijective, hence $V_i^{U_m}$ is perfect. This shows $V_i^{U_m}$ is perfectoid. As V_i^+ is integrally closed in V_i and $V_i^{+U_m} = V_i^{U_m} \cap V_i^+$, $V_i^{+U_m}$ is also integrally closed in $V_i^{U_m}$, whence the assertion. □

Remark 2.17. Notice that although we constructed U_m-invariant elements $N_{U_m}(u_i)$ and $N_{U_m}(y_{ij})$ in V_i, the proof of perfectoidness of (V^{U_m}, V^{+U_m}) above did not really make use of $N_{U_m}(y_{ij})$. It would be ideal if we actually have

$$V^{U_m} = \mathbb{F}_q(\!(N_{U_m}(u_i)^{1/p^\infty})\!)\langle N_{U_m}(y_{ij})^{1/p^\infty}, j \neq i\rangle$$
$$V^{+U_m} = \mathbb{F}_q[\![N_{U_m}(u_i)^{1/p^\infty}]\!]\langle N_{U_m}(y_{ij})^{1/p^\infty}, j \neq i\rangle,$$

but we do not know whether this is true at the moment.

3. Projectivized Banach–Colmez spaces

In this section, we show the projectivization of an absolute Banach–Colmez space $\mathbf{B}^{\varphi^n=\pi^d}$ of slope < 1 is perfectoid. We assume $0 < d < n$ throughout this section.

Notation 3.1. To ease the notation, we may sometimes supress the index $0 \leq i \leq d-1$. Thus for the chart $\mathcal{W}_i = \mathrm{Spa}(W_i, W_i^+)$ of $\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\}$ and the chart $\mathcal{V}_i = \mathrm{Spa}(V_i, V_i^+)$ of the tame quotient, we simply write them respectively as

$$\mathcal{W} = \mathrm{Spa}(W, W^+) = \mathrm{Spa}\,\mathbb{F}_q(\!(x^{1/p^\infty})\!)[\![y_j^{1/p^\infty}]\!], \quad \mathcal{V} = (V, V^+) = \mathrm{Spa}\,\mathbb{F}_q(\!(u^{1/p^\infty})\!)[\![y_j^{1/p^\infty}]\!].$$

When it is clear, we may also just write y for y_j. Such a convention is harmless, since all the constructions and computations below apply verbatim for every i. We shall restore the indices only when comparing different charts or their intersections.

Motivated by [Han16, Theorem 1.2], we give the following criterion for a Tate–Huber pair to be a finite étale torsor over its quotient by a finite group.

Proposition 3.2. *Let (A, A^+) be a complete Tate-Huber pair with a continuous action of a finite group G and put $(B, B^+) = (A^G, A^{+G})$. Suppose there exists $a \in A$ such that $g(a) - a \in A^\times$ for every nontrivial $g \in G$, then (B, B^+) is a complete Tate-Huber pair with $A = B[a]$ and A/B is a finite étale G-torsor.*

The idea is that the elements $g(a)$, for $g \in G$, acts as $|G|$ distinct points if $g(a)-a \in A^\times$. Thus the condition separates the extension A/B.

Proof. The pair (B, B^+) being a complete Tate–Huber pair is standard, so we focus on the rest of the claim. For such an element $a \in A$, denote $g(a)$ by a_g. Then since $g(a)-a \in A^\times$ for every nontrivial $g \in G$, we have $a_g - a_h \in A^\times$ for $g \neq h$. For each $g \in G$, consider

$$L_g(T) := \prod_{\substack{h \in G \\ h \neq g}} \frac{T - a_h}{a_g - a_h} \in A[T].$$

We have $L_g(a_h) = 1$ if $g = h$ and $L_g(a_h) = 0$ if $g \neq h$. The group action permutes these polynomials by $h(L_g(T)) = L_{hg}(T)$ for $h, g \in G$. For $x \in A$, set

$$H_x(T) := \sum_{g \in G} g(x) L_g(T).$$

Since G permutes its factors, we get $H_x(T) \in B[T]$. As $H_x(a) = x$, this proves $A = B[a]$.

Now consider

$$P(T) := \prod_{g \in G} (T - a_g) \in B[T].$$

Since $P(a) = 0$ and $A = B[a]$, evaluation at a gives a surjection from $B[T]/(P)$ to A. We show it is also injective. Suppose we have $F(T) \in B[T]$ of degree less than $|G|$ and satisfies $F(a) = 0$. Since its coefficients are G-invariant, we also get $F(a_g) = 0$ for any $g \in G$. Since the differences $a_g - a_h$ belong to $A^\times$ for $g \neq h$ by assumption, successive divion by $T - a_g$ shows $P(T)$ divides $F(T)$. As P is monic and $\deg(F) < |G|$, this forces $F(T) = 0$. Thus we get $A \cong B[T]/(P(T))$. Since we also have

$$P'(a) = \prod_{g \neq 1} (a - a_g) \in A^\times,$$

the extension A/B is finite étale G-torsor. □

We now construct such extension-separating element for each $V^{U_{m+1}}$ over V^{U_m}.

Lemma 3.3. *For every $m \geq 1$, there exists $\theta_m \in V^{U_{m+1}}$ such that*

$$g(\theta_m) - \theta_m \in (V^{U_{m+1}})^\times$$

for every nontrivial $g \in U_m/U_{m+1}$.

Proof. Suppose $V = V_i$ and write $m = cd + f$ with $0 \leq f < d$. Consider the d-dimensional formal variable $X = (x_0, \ldots, x_{d-1})$. We set $Z = \pi^{-f} X$ and let ξ_i be the i-th coordinate of Z. More explicitely, using the description of π-action given in Lemma 2.1, we have

$$\xi_i = \begin{cases} x_{i+f}^{1/q^f} & \text{if } i + f < d \\ x_{i+f-d}^{1/q^{n-d+f}} & \text{if } i + f \geq d \end{cases} \tag{3.1}$$

We then define

$$\theta_m := \frac{N_{U_{m+1}}(\xi_i)}{N_{U_m}(x_i)^{q^{cn}}}.$$

By Proposition 2.14, θ_m is U_{m+1}-invariant. Since μ_{q-1} acts on the two orbit norms by the same scalar, we also get $\theta_m \in V_i^{U_{m+1}}$.

Consider an element $g = 1 + \pi^m b \in U_m$ with $b \in \mathcal{O}_K^\times$. We first compute the action of g on ξ_i. By (3.1), we can choose $0 < \lambda \leq 1$ in $\mathbb{Z}[1/p]$ such that $\xi_j \in x_i^\lambda W_i^+$ for every j. Then using the formal $\mathcal{O}_K$-module law, we have

$$g(Z) = F(Z, b\pi^{cd}(X)) = F(Z, b(X^{cn})).$$

Since $b(x_j) - \bar{b}(x_j) \in x_i^q W_i^+$ by Lemma 2.1, we get

$$g(\xi_i) - \xi_i - \bar{b}(x_i^{q^{cn}}) \in x_i^{q^{cn}+\lambda} W_i^+. \tag{3.2}$$

We now compute the difference between ξ_i and its U_{m+1}-orbit norm. For $r \geq 0$, consider $h = 1 + \pi^{m+1+r} b' \in U_{m+1+r}$, then we have

$$h(Z) = F(Z, Y), \quad Y = b'\pi^{cd+r+1}(X).$$

By Lemma 2.6, every component of Y belongs to $x_i^{1+(q-1)Q_{cd+r+1}} W_i^+$. As every mixed term of $F(Z, Y)$ contains a Y-component, we then get

$$h(\xi_i) - \xi_i \in x_i^{1+(q-1)Q_{cd+r+1}} W_i^+.$$

By Lemma 2.7, as $r \to \infty$, we have

$$\frac{1 + (q-1)Q_{cd+r+1} - \lambda}{q^{r+1}} \to \infty;$$

direct comparison also shows $1 + (q-1)Q_{cd+r+1} \geq q^{cn+r+1}$. Thus applying Lemma 2.12, we get

$$N_{U_{m+1}}(\xi_i) - \xi_i \in x_i^{q^{cn}+\lambda(1-1/q)} W_i^+. \tag{3.3}$$

Finally, since $N_{U_m}(x_i)$ is U_m-invariant, (3.2) and (3.3) together give

$$g(\theta_m) - \theta_m = \left(\frac{x_i}{N_{U_m}(x_i)}\right)^{q^{cn}} (\bar{b} + \epsilon)$$

for some $\epsilon \in W_i^{\circ\circ}$. Thus $g(\theta_m) - \theta_m$ is a unit in W_i; it also lies in $V_i^{U_{m+1}}$ because θ_m lies in $V_i^{U_{m+1}}$. Hence $g(\theta_m) - \theta_m$ is a unit in $V_i^{U_{m+1}}$. □

For $m \geq 1$, put $(R_m, R_m^+) = (V^{U_m}, V^{+U_m})$ and let $\mathcal{R}_m = \mathrm{Spa}(R_m, R_m^+)$. Such $\mathcal{R}_m$ form a tower of affinoid perfectoid spaces.

Proposition 3.4. *The natural maps $R_m \to R_{m+1}$ induce a tower of finite étale torsors*

$$\cdots \to \mathcal{R}_{m+1} \to \mathcal{R}_m \to \cdots \to \mathcal{R}_1$$

where $\mathcal{R}_n \to \mathcal{R}_m$ is a finite étale U_m/U_n-torsor for every $1 \leq m < n$.

Proof. Using the elements θ_m constructed in Lemma 3.3, the criterion from Lemma 3.2 implies that R_{m+1} is a finite étale U_m/U_{m+1}-torsor over R_m. Since R_{m+1}^+ is the integral closure of R_m^+ in R_{m+1}, geometrically $\mathcal{R}_{m+1} \to \mathcal{R}_m$ is also a finite étale U_m/U_{m+1}-torsor (cf. [Sch26, Definition 6.2]) with Čech groupoid given by

$$(U_m/U_{m+1}) \times \mathcal{R}_{m+1} \rightrightarrows \mathcal{R}_{m+1}.$$

Since U_m are normal subgroups of U_1 and all the actions are induced by the same U_1-action, induction gives the general case for $m < n$. □

Lemma 3.5. *$\bigcup_{m\geq 1} R_m^+$ is dense in V^+ and $\bigcup_{m\geq 1} R_m$ is dense in V.*

Proof. By Proposition 2.14, we have

$$\frac{N_{U_m}(u)}{u} - 1, N_{U_m}(y) - y \in u^{Q_m/q} V^+,$$

where, recall from Notation 2.5, for $m = cd + f$ with $0 \leq f < d$ we set $Q_m = (q^{cn+f} - 1)/(q-1)$. Since $Q_m \to \infty$ as $m \to \infty$, the orbit norms $N_{U_m}(u)$ and $U_{U_m}(y)$ converge to u and y respectively. Similarly, for every $\ell \geq 0$, the p^ℓ-th roots of $N_{U_m}(u)$ and $N_{U_m}(y)$ also converge to the p^ℓ-th roots of u and y respectively; such p-power roots also belong to R_m^+ by part (ii) of Corollary 2.15. Therefore, we get

$$\overline{\bigcup_{m \geq 1} R_m^+} \supseteq \mathbb{F}_q[u^{1/p^\infty}, y^{1/p^\infty}], \tag{3.4}$$

and the latter is dense in V^+. This proves the first claim. Now part (i) of Corollary 2.15 gives $N_{U_1}(u) \in R_1^\times$, hence we get $N_{U_1}(u)/u \in (V^+)^\times$ and $V = V^+[1/N_{U_1}(u)]$. As $N_{U_1}(u)^{-1} \in R_1 \subset R_m$, multiplying (3.4) by powers of $N_{U_1}(u)^{-1}$ proves the density of $\bigcup_{m \geq 1} R_m$ in V. □

Combining this density lemma with the tower of finite étale torsors, we now upgrade the ring-level perfectoidness of Proposition 2.16 into a geometric one.

Proposition 3.6. *The natural morphism $\mathcal{V} \to \mathcal{R}_1$ is a pro-étale U_1-torsor. Consequently, we have an isomorphisms of v-sheaves*

$$\mathcal{W}/\underline{\mathcal{O}_K^\times} \cong \mathcal{V}/\underline{U_1} \cong \mathcal{R}_1.$$

In particular, $\mathcal{W}/\underline{\mathcal{O}_K^\times}$ is affinoid perfectoid.

Proof. Let $t = N_{U_1}(u)$. By part (i) of Corollary 2.15, t is a topologically nilpotent unit in R_m for every $m \geq 1$. Thus for every $s \geq 0$, we have

$$R_m \cap t^s V^+ = t^s R_m^+,$$

hence V^+ is identified with the t-adic completion of $\varinjlim_m R_m^+$ by Lemma 3.5. Also since $V = V^+[1/t]$ and each $\mathcal{R}_m$ is affinoid perfectoid (Proposition 2.16), by [Sch26, Proposition 6.5] we get an isomorphism of perfectoid spaces

$$\mathcal{V} \cong \varprojlim_m \mathcal{R}_m.$$

By Proposition 3.4, such $\mathcal{R}_m$ form a tower of finite étale torsors and we have

$$\begin{aligned} \mathcal{V} \times_{\mathcal{R}_1} \mathcal{V} &\cong \varprojlim_m (\mathcal{R}_m \times_{\mathcal{R}_1} \mathcal{R}_m) \\ &\cong \varprojlim_m (\underline{U_1/U_m} \times \mathcal{R}_m) \\ &\cong \underline{U_1} \times \mathcal{V}. \end{aligned}$$

Therefore, $\mathcal{V} \to \mathcal{R}_1$ is a pro-étale U_1-torsor, hence we get

$$\mathcal{R}_1 \cong \mathrm{Coeq}(\underline{U_1} \times \mathcal{V} \rightrightarrows \mathcal{V}) = \mathcal{V}/\underline{U_1}.$$

Finally, part (ii) of Proposition 2.4 gives $\mathcal{W}/\underline{\mu_{q-1}} \cong \mathcal{V}$ where the μ_{q-1}-action commutes with the U_1-action. Thus we get $\mathcal{W}/\underline{\mathcal{O}_K^\times} \cong \mathcal{R}_1$, which is affinoid perfectoid. □

Theorem 3.7. *For any $0 < d < n$, the v-sheaf quotient $\mathbf{B}^{\varphi^n = \pi^d} \setminus \{0\}/\underline{\mathcal{O}_K^\times}$ is representable by a perfectoid space.*

Proof. By Proposition 2.3, the affinoid perfectoid cover $\bigcup_{i=0}^{d-1} \mathcal{W}_i$ of $\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\}$ is $\mathcal{O}_K^\times$-stable, thus we get a commutative diagram

$$\begin{tikzcd}
\mathcal{W}_i \arrow[r] \arrow[d] & \mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\} \arrow[d] \\
\mathcal{W}_i/\underline{\mathcal{O}_K^\times} \arrow[r] & (\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\})/\underline{\mathcal{O}_K^\times}
\end{tikzcd}$$

As the upper horizontal map is an open immersion and the quotient map on the right is v-surjective, the lower horizontal map is also an open immersion. As such $\mathcal{W}_i/\underline{\mathcal{O}_K^\times}$ cover $(\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\})/\underline{\mathcal{O}_K^\times}$ and each of them is perfectoid by Proposition 3.6, $(\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\})/\underline{\mathcal{O}_K^\times}$ is covered by open perfectoid subspaces, hence perfectoid. $\square$

Corollary 3.8. *For any $0 < d < n$, the Grassmannian $\mathrm{Gr}_{1,C}(\mathbf{B}^{\varphi^n=\pi^d})$ is representable by a perfectoid space.*

Proof. By definition, we have

$$\begin{aligned}
\mathrm{Gr}_{1,C}(\mathbf{B}^{\varphi^n=\pi^d}) &= ((\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\}) \times_{\mathrm{Spd}\,\mathbb{F}_q} \mathrm{Spd}\,C)/\underline{K^\times} \\
&= ((\mathbf{B}^{\varphi^n=\pi^d}\setminus\{0\})/\underline{\mathcal{O}_K^\times} \times_{\mathrm{Spd}\,\mathbb{F}_q} \mathrm{Spd}\,C)/(\varphi\times\mathrm{id})^{\mathbb{Z}}.
\end{aligned}$$

Using Theorem 3.7, the product before taking the quotient by $\varphi\times\mathrm{id}$ is perfectoid. Since the action of $\varphi\times\mathrm{id}$ is properly discontinuous on the product, the quotient is perfectoid. $\square$

Remark 3.9. Recall from section 1.3, for $S \in \mathrm{Perf}_C$, the moduli space $W_{n,1}$ parametrizes surjections of the form $\mathcal{O}(1/n) \twoheadrightarrow \mathcal{E}$ over $X_{S^\flat}$ with $\mathcal{E}$ isomorphic to $\mathcal{O}(1)$ at every geometric point. By duality, a surjection $\mathcal{O}(1/n) \twoheadrightarrow \mathcal{O}(1)$ is equivalent to a stably injective morphism $\mathcal{O} \hookrightarrow \mathcal{O}(\frac{n-1}{n})$; also notice both $\mathcal{O}(1)$ and $\mathcal{O}$ have automorphism group $K^\times$. Therefore, we have

$$\begin{aligned}
W_{n,1} &\cong \left(\mathcal{I}\mathrm{nj}\left(\mathcal{O}, \mathcal{O}\left(\frac{n-1}{n}\right)\right) \times_{\mathrm{Spd}\,\mathbb{F}_q} \mathrm{Spd}\,C\right)/\underline{K^\times} \\
&= \mathrm{Gr}_{1,C}(\mathbf{B}^{\varphi^n=\pi^{n-1}}).
\end{aligned}$$

Thus Corollary 3.8 gives an alternative local proof of Theorem 1.4.

4. Plücker morphisms

For classical Grassmannians of a vector sapce E, there is the Plücker map which embeds the k-Grassmannian of E into the projective space of the k-th exterior power of E. In this section, we study the analogous Plücker morphisms between Grassmannians of absolute BC spaces. As BC spaces are characterized by both their rank and degree, the latter will also play a role in describing the image of these Plücker morphisms.

4.1. **The classical Plücker embedding.** To motivate our later constructions, we give a brief exposition on classical Plücker map and Plücker relations. See [Bou70, § III.11-13] for more details on this material.

Let E be an n-dimensional vector space over a field K and let $0 < k < n$. For a k-dimensional subspace W of E, the *Plücker map* is given by

$$\begin{aligned} \mathrm{Pl} : \mathrm{Gr}_k(E) &\to \mathbb{P}(\bigwedge^k E) \\ W &\mapsto \bigwedge^k W. \end{aligned}$$

Notice here that changing the basis of W would change the wedge product $\bigwedge^k W$ by a scalar. A vector in $\mathbb{P}(\bigwedge^k E)$ is called *decomposable* if it is a single wedge product of k vectors in E, and the image of the Plücker map is exactly this decomposable locus inside the projective space.

We now give a more algebraic description of the decomposable vectors. Denote by E^* the dual vector space of E and let

$$\langle -, - \rangle_r : \bigwedge^r E^* \otimes \bigwedge^r E \to K$$

be the perfect pairing induced by the evaluation map. The adjoint of this map then gives the *contraction map*

$$c : \bigwedge^{k-1} E^* \otimes \bigwedge^k E \to E$$

which satisfies $\langle \beta^*, c(\alpha^*, w) \rangle_1 = \langle \alpha^* \wedge \beta^*, w \rangle_r$. More explicitely, given $\alpha^* \in \bigwedge^{k-1} E^*$ and $w_1 \wedge \cdots \wedge w_k \in \bigwedge^k E$, we have

$$c(\alpha^*, w_1 \wedge \cdots \wedge w_k) = \sum_{j=1}^{k} (-1)^{k-j} \langle \alpha^*, w_1 \wedge \cdots \wedge \widehat{w_j} \wedge \cdots \wedge w_k \rangle_{k-1} w_j. \tag{4.1}$$

For a nonzero vector $w \in \bigwedge^k E$, consider its associated contraction map and exterior multiplication map:

$$c_w : \bigwedge^{k-1} E^* \to E, \qquad \alpha^* \mapsto c(\alpha^*, w) \tag{4.2}$$

$$m_w : E \to \bigwedge^{k+1} E, \qquad v \mapsto v \wedge w. \tag{4.3}$$

If w belongs to the image of the Plücker map, these two maps describe the k-plane W associated to w in two directions: the image of the contraction map c_w recovers the plane $W \subseteq E$, and the the exterior multiplication map m_w vanishes exactly on W. Therefore, we have:

Proposition 4.1 ([Bou70, Proposition III.13.16])**.** *A nonzero vector $w \in \bigwedge^k E$ is decomposable if and only if $m_w \circ c_w = 0$.*

The image of the Plücker map can also be described via *Plücker relations*. To recover such quadratic equations, consider

$$\begin{aligned} Q : \bigwedge^k E &\to \mathrm{Hom}_K(\bigwedge^{k-1} E^*, \bigwedge^{k+1} E) \\ w &\mapsto m_w \circ c_w. \end{aligned}$$

Let $\{e_i\}$ be a basis of E with dual basis $\{e_i^*\}$. For a vector $w \in \bigwedge^k E$, write $w = \sum_I p_I e_I$ for ordered k-tuples $I \subseteq \{1, \ldots, n\}$, where e_I are the basis vectors of $\bigwedge^k E$ and p_I are the

Plücker coordinates. Then to have $Q(w) = 0$, we need $Q(w)(e_J^*) = 0$ for every dual basis vector e_J^* of $\bigwedge^{k-1} E^*$, which means the coordinate componenets of $Q(w)$ all vanish. This gives the Plücker relations.

Example 4.2. Let E be a 4-dimensional vector space with basis $e_1, \dots, e_4$ and consider the Plücker map

$$\mathrm{Pl} : \mathrm{Gr}_2(E) \to \mathbb{P}(\bigwedge^2 E).$$

By the discussion above, for $w = \sum_I p_I e_I \in \bigwedge^2 E$, the class $[w]$ is in the image of Pl if and only if $Q(w)(e_i^*) = 0$ for $i = 1, 2, 3, 4$. From (4.1), we get

$$c_w(e_1^*) = p_{12}e_2 + p_{13}e_3 + p_{14}e_4.$$

Taking the wedge product with w then gives $(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})e_{234}$, which induces the well-known Plücker relation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

Replacing e_1^* by other dual basis vectors gives the same quadratic relation in this example.

4.2. **The decomposable locus.** We now transition back to the setting of vector bundles on the Fargues–Fontaine curve, and define the decomposable locus of Grassmannians of absolute Banach–Colmez spaces.

Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$ and let $\mathcal{E}$ be a vector bundle on X_S or rank n. We write $\mathcal{E}^\vee$ for its dual bundle. For any integer $0 < k < n$, we have the contraction map

$$c : \bigwedge^{k-1} \mathcal{E}^\vee \otimes \bigwedge^{k} \mathcal{E} \to \mathcal{E}$$

which is adjoint to the perfect pairing $\langle -, - \rangle : \bigwedge^k \mathcal{E}^\vee \otimes \bigwedge^k \mathcal{E} \to \mathcal{O}_{X_S}$. By construction, we have $\langle \beta, c(\alpha, w) \rangle = \langle \alpha \wedge \beta, w \rangle$ on local sections. By further composing the contraction map c with the exterior product map $m : \mathcal{E} \otimes \bigwedge^k \mathcal{E} \to \bigwedge^{k+1} \mathcal{E}$, we get a morphism

$$m \circ (c \otimes \mathrm{id}) : \bigwedge^{k-1} \mathcal{E}^\vee \otimes \bigwedge^{k} \mathcal{E} \otimes \bigwedge^{k} \mathcal{E} \to \bigwedge^{k+1} \mathcal{E}.$$

Thus on local sections, we have $m \circ (c \otimes \mathrm{id})(\alpha, w_1, w_2) = c(\alpha, w_1) \wedge w_2$. All these constructions commute with base change in S.

Now take $\mathcal{E}$ to be the semistable vector bundle $\mathcal{E}_{n,d}$ of rank n and degree d. Its k-th wedge power gives $\bigwedge^k \mathcal{E}_{n,d} = \mathcal{E}_{N,D}$ where $\mathcal{E}_{N,D}$ is the semistable vector bundle over X_S of rank $N = \binom{n}{k}$ and degree $D = d \cdot \binom{n-1}{k-1}$.

Consider the v-sheaf $\mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d})/\underline{K^\times}$. An S-point of this sheaf is represented by a stably injective morphism $u : \mathcal{L} \to \bigwedge^k \mathcal{E}_{n,d}$ where $\mathcal{L}$ is a line bundle over X_S fibrewise isomorphic to $\mathcal{O}$. For such a pair $(u, \mathcal{L})$, we let

$$c_u := c \circ (\mathrm{id} \otimes u) : \bigwedge^{k-1} \mathcal{E}_{n,d}^\vee \otimes \mathcal{L} \to \mathcal{E}_{n,d}$$

$$m_u := m \circ (\mathrm{id} \otimes u) : \mathcal{E}_{n,d} \otimes \mathcal{L} \to \bigwedge^{k+1} \mathcal{E}_{n,d}.$$

Their composition then gives a morphism

$$q_u := m_u \circ (c_u \otimes \mathrm{id}) : \bigwedge^{k-1} \mathcal{E}_{n,d}^\vee \otimes \mathcal{L}^{\otimes 2} \to \bigwedge^{k+1} \mathcal{E}_{n,d}.$$

Let also $\widetilde{m}_u : \mathcal{E}_{n,d} \to \mathcal{L}^\vee \otimes \bigwedge^{k+1} \mathcal{E}_{n,d}$ be the adjoint morphism of m_u, then we have:

Lemma 4.3. *The morphism q_u is 0 if and only if $\widetilde{m}_u \circ c_u$ is 0.*

Proof. Since $\mathcal{L}$ is a line bundle, adjunction gives an isomorphism

$$\mathrm{Hom}(\bigwedge^{k-1} \mathcal{E}_{n,d}^\vee \otimes \mathcal{L}, \mathcal{L}^\vee \otimes \bigwedge^{k+1} \mathcal{E}_{n,d}) \cong \mathrm{Hom}_{\mathcal{O}_{X_S}}(\bigwedge^{k-1} \mathcal{E}_{n,d}^\vee \otimes \mathcal{L} \otimes \mathcal{L}, \bigwedge^{k+1} \mathcal{E}_{n,d}),$$

under which $\widetilde{m}_u \circ c_u$ corresponds to q_u. This proves the claim. $\square$

Similar to the classical case discussed previously, we now use q_u to define the decomposable locus of $\mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d})/\underline{K^\times}$.

Definition 4.4. For integers $1 \leq k \leq n$ and $d \in \mathbb{N}^*$, we denote by $\mathrm{Dec}^k(\mathcal{E}_{n,d})$ the sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ which assigns to S the set of S-points $[(u, \mathcal{L})]$ of $\mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d})/\underline{K}^\times$ such that $q_u = 0$. We call $\mathrm{Dec}^k(\mathcal{E}_{n,d})$ the *decomposable locus* of $\mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d})/\underline{K}^\times$.

One can easily check that the condition $q_u = 0$ does not depend on the representative $(u, \mathcal{L})$: if $\varphi : (u, \mathcal{L}) \to (u', \mathcal{L}')$ is an isomorphism between two such pairs, then $u' \circ \varphi = u$, hence $q_{u'} \circ (\mathrm{id} \otimes \varphi^{\otimes 2}) = q_u$. If one picks an open subset of X_S and a frame of the vector bundle $\mathcal{E}_{n,d}$, then locally the condition $q_u = 0$ recovers the Plücker relations on such coordinates. Since $\mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d})/\underline{K^\times} = \mathrm{Gr}_1(\mathbf{B}^{\varphi^N = \pi^D})$ for some $N, D \in \mathbb{N}^*$, $\mathrm{Dec}^k(\mathcal{E}_{n,d})$ defines a k-decomposable locus of the Grassmannian.

Lemma 4.5. *The decomposable locus $\mathrm{Dec}^k(\mathcal{E}_{n,d})$ is a v-subsheaf of $\mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d})/\underline{K}^\times$.*

Proof. The maps between vector bundles related to the construction of q_u all commute with base change. Hence for a v-cover $T \to S$, we have $q_{u_T} = q_u|_{X_T}$. Moreover, vanishing of a morphism of vector bundles is local for the v-topology. Therefore, the condition $q_u = 0$ satisfies v-descent. $\square$

4.3. **Plücker morphisms.** In this section, we construct the Plücker morphism between Grassmannians, and identify its image with the decomposable locus defined above.

For $x \in |X_S|$ and $\mathcal{E}$ a vector bundle on X_S, write $\mathcal{E}(x) = \mathcal{E} \otimes_{\mathcal{O}_{X_S}} \kappa(x)$ for its fibre at x. For a morphism f between vector bundles on X_S, let $f(x)$ denote the corresponding map on the fibres.

Lemma 4.6. *Let $f : \mathcal{E} \to \mathcal{F}$ be a morphism of vector bundles on X_S. If the function $x \mapsto \mathrm{rank}_{\kappa(x)} f(x)$ is locally constant, $\ker(f), \mathrm{im}\,(f), \mathrm{coker}(f)$ are vector bundles.*

Proof. The assertion is local on X_S. Since X_S is stably uniform, we may work on an affinoid open $U = \mathrm{Spa}(R, R^+) \subseteq X_S$ with R uniform, on which $\mathcal{E}$ and $\mathcal{F}$ are free and $f(x)$ has constant rank r. Then f comes from a map $f_R : P \to Q$ between finite free R-modules. Let $M = \mathrm{coker}(f_R)$. For every $x \in U$, we have $M \otimes_R \kappa(x) \cong \mathrm{coker}\, f(x)$. Its dimension $\dim \mathcal{F}(x) - r$ is locally constant on U. Hence M is finite projective by [KL15, Proposition 2.8.4]. It follows that $Q \to M$ splits, so $N = \mathrm{im}\,(f_R)$ is finite projective. Thus $P \to N$ also splits, so $K = \ker(f_R)$ is finite projective. Tensoring these split exact sequences with $\mathcal{O}_U$ identifies $\ker(f)|_U, \mathrm{im}\,(f)|_U, \mathrm{coker}(f)|_U$ with the vector bundles associated to K, N, M respectively. $\square$

Lemma 4.7. *Let $n, d, k \in \mathbb{N}^*$ and assume $kd < n$.*

(i) For any injective morphism $i : \mathcal{F} \to \mathcal{E}_{n,d}$ over X_S with $\mathcal{F}$ fibrewise isomorphic to $\mathcal{O}^k$, $\mathrm{coker}(i)$ is a vector bundle on X_S.

(ii) For any injective morphism $u : \mathcal{L} \to \bigwedge^k \mathcal{E}_{n,d}$ over X_S with $\mathcal{L}$ fibrewise isomorphic to $\mathcal{O}$, $\mathrm{coker}(u)$ is a vector bundle on X_S.

Proof. By [JLH21, Remark A.2.3], it suffices to show that $\mathcal{F}$ and $\mathcal{L}$ are saturated when $S = \mathrm{Spa}(F, F^+)$ is a geometric point. For (i), we have

$$\frac{\deg \mathcal{F} + 1}{\text{rank } \mathcal{F}} = \frac{1}{k} > \frac{d}{n} = \mu(\mathcal{E}_{n,d}).$$

Similarly for (ii), we have

$$\frac{\deg \mathcal{L} + 1}{\text{rank } \mathcal{L}} = 1 > \mu(\bigwedge^k \mathcal{E}_{n,d}) = \frac{kd}{n}.$$

By [JLH21, Lemma A.2.1], such inequalities imply both $\mathcal{F}$ and $\mathcal{L}$ are saturated, hence the corresponding quotients are vector bundles on X_S. □

Let $i : \mathcal{F} \to \mathcal{E}_{n,d}$ be a stably injective morphism over X_S with $\mathcal{F}$ fibrewise isomorphic to $\mathcal{O}^k$. Taking its determinant gives $\wedge^k i : \det \mathcal{F} \to \bigwedge^k \mathcal{E}_{n,d}$, which is also stably injective. Moreover, given two such pairs $(i, \mathcal{F})$ and $(i', \mathcal{F}')$ together with an isomorphism $\varphi : (i, \mathcal{F}) \to (i', \mathcal{F}')$, we have $i' \circ \varphi = i$, hence $(\wedge^k i') \circ \det \varphi = \wedge^k i$. Thus $\det(\varphi)$ gives an isomorphism between $(\wedge^i, \det \mathcal{F})$ and $(\wedge^k i', \det \mathcal{F}')$. Therefore, we have a well-defined morphism of v-sheaves

$$\begin{aligned} \mathrm{Pl} : \mathcal{I}\mathrm{nj}(\mathcal{O}^k, \mathcal{E}_{n,d})/\underline{\mathrm{GL}_k(K)} &\to \mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d})/\underline{K^\times} \\ [(i, \mathcal{F})] &\mapsto [(\wedge^k i, \det \mathcal{F})] \end{aligned}$$

which we call the *Plücker morphism*.

Our motivation for considering such a map comes from the special case when $d = 1$ and $k = n-1$, where the quotient on the right hand side is isomorphic to $\mathcal{I}\mathrm{nj}(\mathcal{O}, \mathcal{O}(\frac{n-1}{n}))/\underline{K^\times}$ and the left one is isomorphic to $\mathcal{S}\mathrm{urj}(\mathcal{O}(1/n), \mathcal{O}(1))/\underline{K^\times}$. Duality then implies that Pl is an isomorphism (cf. Remark 3.9). In the general case, however, such a Plücker morphism need not be surjective. We will show later that it nevertheless induces an isomorphism onto the decomposable locus.

Lemma 4.8. *The Plücker morphism*

$$\mathrm{Pl} : \mathcal{I}\mathrm{nj}(\mathcal{O}^k, \mathcal{E}_{n,d})/\underline{\mathrm{GL}_k(K)} \to \mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d})/\underline{K^\times}$$

factors through the decomposable locus $\mathrm{Dec}^k(\mathcal{E}_{n,d})$.

Proof. Let $(i, \mathcal{F})$ represent an S-point of $\mathcal{I}\mathrm{nj}(\mathcal{O}^k, \mathcal{E}_{n,d})/\underline{\mathrm{GL}_k(K)}$. We choose local generators $v_1, \ldots, v_k$ of $\mathcal{F}$. Under the Plücker morphism, the corresponding local generator of $\det \mathcal{F}$ is given by $z = i(v_1) \wedge \cdots \wedge i(v_k)$. By definition of the contraction map, for every local section α of $\bigwedge^{k-1} \mathcal{E}_{n,d}^\vee$ the contraction $c(\alpha, z)$ belongs to the submodule generated by $i(v_1), \ldots, i(v_k)$. Therefore, we get $c(\alpha, z) \wedge z = 0$ and $q_{\wedge^k i} = 0$. Hence the image of Pl lies in $\mathrm{Dec}^k(\mathcal{E}_{n,d})$. □

Proposition 4.9. *Assume $kd < n$. For every pair $(u, \mathcal{L})$ representing an S-point of* $\mathrm{Dec}^k(\mathcal{E}_{n,d})$*, there is a unique rank k subbundle $\mathcal{V}_u \subseteq \mathcal{E}_{n,d}$ such that*

$$\det \mathcal{V}_u = u(\mathcal{L}) \subseteq \bigwedge^k \mathcal{E}_{n,d}.$$

Proof. By part (ii) of Lemma 4.7, $u(\mathcal{L}) \subseteq \bigwedge^k \mathcal{E}_{n,d}$ is a line subbundle. Since $q_u = 0$, Proposition 4.1 and Lemma 4.3 show that $u(x)(\mathcal{L}(x))$ is decomposable for every $x \in |X_S|$, hence it is a single wedge product of k vectors. Therefore, $\ker(\widetilde{m}_u(x))$ has dimension k and $\widetilde{m}_u(x)$ has rank $n-k$. Lemma 4.6 then gives a rank k subbundle

$$\mathcal{V}_u := \ker(\widetilde{m}_u) \subseteq \mathcal{E}_{n,d}.$$

Specifically, we get $\mathcal{V}_u(x) = \ker(\widetilde{m}_u(x))$, hence $\det(\mathcal{V}_u(x)) = u(x)(\mathcal{L}(x))$. Therefore, the composition

$$\mathcal{L} \xrightarrow{u} \bigwedge^k \mathcal{E}_{n,d} \to (\bigwedge^k \mathcal{E}_{n,d})/\det \mathcal{V}_u$$

is zero at every fibre, hence the composition itself is also zero by Lemma 4.6. Thus the morphism $u : \mathcal{L} \to \bigwedge^k \mathcal{E}_{n,d}$ factors through $\det \mathcal{V}_u$ and gives $u(\mathcal{L}) = \det \mathcal{V}_u$.

Suppose $\mathcal{V} \subseteq \mathcal{E}_{n,d}$ is another rank k subbundle with $\det \mathcal{V} = u(\mathcal{L})$. We then have $\mathcal{V} \wedge u(\mathcal{L}) = 0$, which gives $\mathcal{W} \subseteq \ker(\widetilde{m}_u) = \mathcal{V}_u$. Since $\det \mathcal{V} = u(\mathcal{L}) = \det \mathcal{V}_u$, we get $\mathcal{V} = \mathcal{V}_u$, which proves the uniqueness part. $\square$

Lemma 4.10. *Let $\mathcal{V}_u$ be the subbundle of Proposition 4.9. For every geometric point $\overline{x}$ of S, we have $\mathcal{V}_{u,\overline{x}} \cong \mathcal{O}^k_{X_{\overline{x}}}$.*

Proof. Since $\det(\mathcal{V}_u) = \mathcal{L}$, the vector bundle $\mathcal{V}_u$ has degree 0. Let $\overline{x}$ be any geometric point of S. Suppose the maximal HN piece of $\mathcal{V}_{u,\overline{x}}$ has positive degree and rank $r \le k$, then its slope is at least $1/r$. Since $\mathcal{E}_{n,d}$ is semistable, we get $1/r \le d/n$, and the latter is $< 1/k$ by assumption; this contradicts the constraint $r \le k$. Thus all the HN slopes of $\mathcal{V}_{u,\overline{x}}$ are non-positive, which implies $\mathcal{V}_{u,\overline{x}} \cong \mathcal{O}^k_{X_{\overline{x}}}$. $\square$

Theorem 4.11. *Suppose $kd < n$. The Plücker morphism induces an isomorphism*

$$\mathrm{Pl} : \mathrm{Gr}_k(\mathbf{B}^{\varphi^n = \pi^d}) \xrightarrow{\sim} \mathrm{Dec}^k(\mathcal{E}_{n,d}).$$

Proof. It suffices to show that for every $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathrm{Pl}(S)$ is bijective. Surjectivity is given by Proposition 4.9 and Lemma 4.10. If two S-points $(i, \mathcal{F})$ and $(i', \mathcal{F}')$ of the Grassmannian have the same image under $\mathrm{Pl}(S)$, then we get $\det \mathcal{F} \cong \det \mathcal{F}'$, and the images $(\wedge^k i)(\mathcal{F}), (\wedge^k i')(\mathcal{F}')$ identify the same line subbundle of $\bigwedge^k \mathcal{E}_{n,d}$. By the uniqueness part of Proposition 4.9, $i(\mathcal{F})$ and $i'(\mathcal{F}')$ then identify the same vector subbundle of $\mathcal{E}_{n,d}$, hence they represent the same isomorphism class in the Grassmannian. $\square$

We now show the decomposable locus is Zariski closed inside $\mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d})/\underline{K^\times}$. First we recall the following lemma of Fargues–Scholze:

Lemma 4.12 ([FS24, Lemma IV.4.23])**.** *Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$ be an affinoid perfectoid space with pseudouniformizer ϖ and $I \subseteq (0, \infty)$ a compact interval with rational ends. Then for every ideal $J \subseteq \mathcal{O}(Y_{S,I})$, there exists a unique Zariski closed perfectoid subspace $S(J)$ of S such that for every perfectoid space $T \to S$, its structural morphism factors through $S(J)$ if and only if the image of J is 0 along $\mathcal{O}(Y_{S,I}) \to \mathcal{O}(Y_{T,I})$.*

Proposition 4.13. *The decomposable locus*

$$\mathrm{Dec}^k(\mathcal{E}_{n,d}) \subseteq \mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d})/\underline{K^\times}$$

is Zariski closed.

Proof. Let us simply write $\mathcal{I}$ for $\mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d})/\underline{K^\times}$. For any affinoid perfectoid $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$, we want to show the fibre product

$$\begin{array}{ccc} S \times_{\mathcal{I}} \mathrm{Dec}^k(\mathcal{E}_{n,d}) & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathrm{Dec}^k(\mathcal{E}_{n,d}) & \longrightarrow & \mathcal{I} \end{array}$$

is Zariski closed in S. Since the decomposable locus is defined by the condition $q_u = 0$, this fibre product parametrizes exactly those $T \to S$ such that the pullback $(q_u)_T$ is 0.

For $(u, \mathcal{L})$ representing an S-point of $\mathcal{I}$, the associated map q_u gives a global section of

$$\mathcal{H} := \mathcal{H}\mathrm{om}_{\mathcal{O}_{X_S}}(\bigwedge^{k-1} \mathcal{E}_{n,d}^\vee \otimes \mathcal{L}^{\otimes 2}, \bigwedge^{k+1} \mathcal{E}_{n,d})$$

over X_S. We pick the interval $I = [1, q] \subset (0, \infty)$ and consider $Y_{S,I}$. Also denote by $B_{S,I} = \mathcal{O}(Y_{S,I})$ and by $\pi : Y_{S,I} \to X_S$ the covering map. Since $Y_{S,I}$ is sousperfectoid, [SW20, Theorem 5.2.8, Proposition 6.3.4] imply that $H := \Gamma(Y_{S,I}, \pi^*\mathcal{H})$ is a finite projective $B_{S,I}$-module. We let h_u be the pullback of q_u along the cover π. Then evaluation at h_u gives a $B_{S,I}$-linear map $H^\vee \to B_{S,I}$, and

$$J := \mathrm{im}\,(H^\vee \to B_{S,I})$$

gives an ideal of $B_{S,I}$. As H is finite projective, $J = 0$ is equivalent to $h_u = 0$.

By Lemma 4.12, the ideal J gives a Zariski closed perfectoid subspace, say S^{dec}, of S such that a perfectoid space $T \to S$ factors through S^{dec} if and only if $J \cdot B_{T,I} = 0$. Assume T is affinoid perfectoid. Since H is finite projective, $J \cdot B_{T,I} = 0$ is equivalent to $h_u \otimes 1 = 0$ in $H \otimes_{B_{S,I}} B_{T,I}$; in the meantime, $h_u \otimes 1$ is exactly the pullback of $(q_u)_T$ to $Y_{T,I}$. Since $I = [1, q]$, Frobenius translates of $Y_{T,I}$ cover Y_T, hence the cover $Y_T \to X_T$ is Frobenius equivariant. Therefore, $J \cdot B_{T,I} = 0$ is equivalent to $(q_u)_T = 0$. This also extends to arbitrary perfectoid $T \to S$ by taking affinoid open cover of Y. Thus by the Yoneda lemma, we get $S^{\mathrm{dec}} \cong S \times_{\mathcal{I}} \mathrm{Dec}^k(\mathcal{E}_{n,d})$, hence the assertion. $\square$

For an absolute BC space of slope $< 1/k$, the Plücker morphism therefore reduces the perfectoidness of its k-Grassmannian to the case of a projectivized absolute BC space. Combining this with what we established in section 3 gives:

Theorem 4.14. *The Grassmannian* $\mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d})$ *is perfectoid if* $kd < n$.

Proof. Let $N = \binom{n}{k}$ and $D = d \cdot \binom{n-1}{k-1}$. Then we have

$$(\mathcal{I}\mathrm{nj}(\mathcal{O}, \bigwedge^k \mathcal{E}_{n,d}) \times_{\mathrm{Spd}\,\mathbb{F}_q} \mathrm{Spd}\,C)/\underline{K^\times} \cong \mathrm{Gr}_{1,C}(\mathbf{B}^{\varphi^N=\pi^D}).$$

The condition $kd < n$ implies $D < N$. Hence Corollary 3.8 shows $\mathrm{Gr}_{1,C}(\mathbf{B}^{\varphi^N=\pi^D})$ is perfectoid. By Theorem 4.11 and Proposition 4.13, the Plücker morphism

$$\mathrm{Pl} : \mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d}) \to \mathrm{Gr}_{1,C}(\mathbf{B}^{\varphi^N=\pi^D})$$

is a Zariski closed immersion, thus $\mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d})$ is perfectoid. $\square$

From the statements leading to Theorem 4.14, one can see that the assumption $kd < n$ is crucial. For $k = 1$, the condition $d < n$ was essential for the convergence of the orbit norms. For higher Grassmannians, the Plücker morphism will not be an isomorphism onto the decomposable locus without assuming $kd < n$.

5. Non-perfectoid Grassmannians

In this section, we show the previous assumption of $kd < n$ is necessary by showing the k-Grassmannian of an absolute BC space is not perfectoid whenever its slope is $\geq 1/k$.

We start with the simple yet foundational example of $\mathrm{Gr}_{1,C}(\mathbf{B}^{\varphi=\pi})$. Recall that

$$\mathcal{I}\mathrm{nj}(\mathcal{O}, \mathcal{O}(1))/\underline{K^\times} \cong (\mathbf{B}^{\varphi=\pi} \setminus \{0\})/\underline{K^\times}.$$

The right hand side is identified with Div^1 by [Far20], and the same argument as in Theorem 1.5 shows the left hand side is isomorphic to $\mathrm{Spd}\, K/\varphi_K^{\mathbb{Z}}$. Hence we have

$$\mathrm{Gr}_{1,C}(\mathbf{B}^{\varphi=\pi}) = (\mathrm{Spd}\, K \times_{\mathrm{Spd}\, \mathbb{F}_q} \mathrm{Spd}\, C)/(\varphi_K \times \mathrm{id})^{\mathbb{Z}} \cong \mathrm{Div}^1_C\,.$$

That this is not perfectoid is presumably well known to the experts; see e.g. [Wei17, p.2]. We nevertheless include a proof here.

Proposition 5.1. *The diamond Div^1_C is not representable by a perfectoid space.*

Proof. Write Div^1_C as $(\mathrm{Spd}\, K \times_{\mathrm{Spd}\, \mathbb{F}_q} \mathrm{Spd}\, C)/(\varphi_K \times \mathrm{id})^{\mathbb{Z}}$. By [FS24, Proposition II.1.17], there is a natural isomorphism $Y^\diamond_{C^\flat} \cong \mathrm{Spd}\, K \times_{\mathrm{Spd}\, \mathbb{F}_q} \mathrm{Spd}\, C$, which also induces $X^\diamond_{C^\flat} \cong (\mathrm{Spd}\, K \times_{\mathrm{Spd}\, \mathbb{F}_q} \mathrm{Spd}\, C)/(\mathrm{id} \times \varphi_{C^\flat})^{\mathbb{Z}}$.[3] For any rational interval $I \subseteq (0, \infty)$, consider the affinoid open subset $Y_{C^\flat, I} = \mathrm{Spa}(B_I, B_I^+)$ of $Y_{C^\flat}$. By [FS24, Proposition II.1.16], there is a radius map $\mathrm{rad} : |Y_{C^\flat}| \to (0, \infty)$ such that $\mathrm{rad} \circ \varphi_{C^\flat} = q \cdot \mathrm{rad}$ and $Y_{C^\flat, I} \subset \mathrm{rad}^{-1}(I)$. As the product of the two Frobenius maps acts trivially on the topological space $|Y_{C^\flat}|$, we get $\varphi_K^n(Y_{C^\flat, I}) \subset \mathrm{rad}^{-1}(q^{-n} I)$. Thus if we take $I = [a, b]$ with $a < b < qa$, the φ_K-translates of I are disjoint, hence the same for the φ_K-translates of $Y^\diamond_{C^\flat, I}$. Hence we get an open immersion

$$Y^\diamond_{C^\flat, I} \to \mathrm{Div}^1_C,$$

and it is enough to show $Y^\diamond_{C^\flat, I}$ is not perfectoid. By [HK26, Remark 11.2 and §13], we are further reduced to show $Y_{C^\flat, I}$ is not perfectoid.

Assume $Y_{C^\flat, I} = \mathrm{Spa}(B_I, B_I^+)$ is perfectoid. Since $Y_{C^\flat, I}$ is affinoid and thus quasicompact, it admits an open cover by finitely many affinoid perfectoid subspaces $U = \mathrm{Spa}(R, R^+)$; specifically, each R is perfectoid.[4] By [FS24, Corollary II.1.12], we can also assume each R is a principal ideal domain. Thus by [Ked19, Corollary 2.9.3], each R is a perfectoid field, hence $Y_{C^\flat, I}$ has finitely many rank-1-points. On the other hand, each $\rho \in [a, b]$ gives rise to a Gauss point y_ρ of $Y_{C^\flat, I}$, which corresponds to a Gauss norm of a perfectoid open unit disk (see the proof of [FS24, Lemma II.1.14]). As such Gauss norms have rank 1 (see [SW20, Example 2.3.7]), we get a contradiction, whence the proposition. □

We now consider the case of higher k-Grassmannians. First we observe the following.

Lemma 5.2. *Assume $0 < k < n$ and $kd \geq n$. For any $S \in \mathrm{Perf}_C$, there exists an exact sequence of vector bundles over the relative Fargues–Fontaine curve $X_{S^\flat}$:*

$$0 \to \mathcal{O}(1/k) \to \mathcal{E}_{n,d} \to \mathcal{E}_{n-k,d-1} \to 0$$

[3] Notice here that Div^1_C and $X^\diamond_{C^\flat}$ correspond to different Frobenius quotients of $Y^\diamond_{C^\flat}$: one is the Frobenius on $\mathrm{Spd}\, K$ and the other is on $\mathrm{Spd}\, C$.

[4] Each U is picked to be affinoid perfectoid, which by definition means R is perfectoid. On the other hand, the assumption of $Y_{C^\flat, I}$ being perfectoid does not imply B_I is perfectoid (see [SW20, Remark 7.1.3]), which is the reason why we need to take such a cover.

Proof. Let us first assume $S = \mathrm{Spa}(C, C^+)$. Assume $kd > n$, then we get $(d-1)/(n-k) > d/n > 1/k$. Thus the slope of $\mathcal{O}(1/k)$ is smaller than that of $\mathcal{E}_{n-k,d-1}$ and the straight Harder–Narasimhan polygon of $\mathcal{E}$ lies below that of $\mathcal{O}(1/k) \oplus \mathcal{E}_{n-k,d-1}$. We then obtain the exact sequence by [BFHHL+22, Theorem 1.1.2]. If $kd = n$, then all three vector bundles have slope $1/k$, hence we have $\mathcal{E}_{n,d} \cong \mathcal{O}(1/k)^{\oplus d}$ and $\mathcal{E}_{n-k,d-1} \cong \mathcal{O}(1/k)^{\oplus(d-1)}$. The splitting of $\mathcal{E}_{n,d}$ then gives the exact sequence. For general $S \in \mathrm{Perf}_C$, the claim follows by pulling back the exact sequence over $X_{C^\flat}$ along $X_{S^\flat} \to X_{C^\flat}$. $\square$

Since the k-Grassmannian of $\mathbf{B}^{\varphi^n=\pi^d}$ parametrizes injections $\mathcal{F} \hookrightarrow \mathcal{E}_{n,d}$ with $\mathcal{F} \cong$ fibrewise isomorphic to $\mathcal{O}^k$, a short exact sequence as above allows us to consider such injections which factor through $\mathcal{O}(1/k)$.

Definition 5.3. For any $k \geq 1$, let $H_k := \mathcal{I}\mathrm{nj}(\mathcal{O}^k, \mathcal{O}(1/k))/\underline{\mathrm{GL}_k(K)}$ and let $H_{k,C} := H_k \times_{\mathrm{Spd}\,\mathbb{F}_q} \mathrm{Spd}\, C$.

Lemma 5.4. *When $kd \geq n$, there exists a Zariski closed immersion $H_{k,C} \to \mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d})$.*

Proof. Using Lemma 5.2, we fix a short exact sequence

$$0 \to \mathcal{O}(1/k) \xrightarrow{\beta} \mathcal{E}_{n,d} \xrightarrow{\gamma} \mathcal{O}((d-1)/(n-k)) \to 0 \tag{5.1}$$

For any affinoid perfectoid $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_C$, we want to show the fibre product

$$\begin{array}{ccc} S \times_{\mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d})} H_{k,C} & \longrightarrow & S \\ \downarrow & & \downarrow \\ H_{k,C} & \longrightarrow & \mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d}) \end{array}$$

is Zariski closed in S. As an S-point of the Grassmannian is a stably injective map $i : \mathcal{F} \to \mathcal{E}_{n,d}$ over $X_{S^\flat}$ with $\mathcal{F} \cong \mathcal{O}^k$ at any geometric point, the fiber product parametrizes such i that factors through $\mathcal{O}(1/k)$, or equivalently that $(\gamma \circ i) = 0$.

Let $I = [1, q] \subset (0, \infty)$ and denote by $\pi : Y_{S^\flat,I} \to X_{S^\flat}$ the covering map. Let $B_{S^\flat,I}$ be the global sections of $Y_{S^\flat,I}$. Via pullback of (5.1) along π, we get a map

$$\pi^*(\gamma \circ i) : F \to N$$

where $F = \Gamma(Y_{S^\flat,I}, \pi^*\mathcal{F})$ and $N = \Gamma(Y_{S^\flat,I}, \pi^*\mathcal{O}(\frac{d-1}{n-k}))$. By [SW20, Theorem 5.2.8, Proposition 6.3.4], both F and N are finite projective $B_{S^\flat,I}$-modules. The map $\pi^*(\gamma \circ i)$ then induces a $B_{S^\flat,I}$-linear evaluation map $F \otimes_{B_{S^\flat,I}} N^* \to B_{S^\flat,I}$, hence its image defines an ideal of $B_{S^\flat,I}$ which we denote by J. By Lemma 4.12, J gives a Zariski closed perfectoid subspace $Z_0 \hookrightarrow S^\flat$ which is universal for perfectoid $T_0 \to S^\flat$ such that $J\mathcal{O}(Y_{T_0,I}) = 0$. By the tilting equivalence $\mathrm{Perf}_S \cong \mathrm{Perf}_{S^\flat}$, such a $Z_0 \hookrightarrow S^\flat$ corresponds to a $Z \hookrightarrow S$, which, by [FS24, Proposition II.0.2], is again a Zariski closed perfectoid subspace. Thus $Z \hookrightarrow S$ is universal for perfectoid $T \to S$ such that $J\mathcal{O}(Y_{T^\flat,I}) = 0$. Assume T is affinoid perfectoid, since F and N are finite projective, $J\mathcal{O}(Y_{T^\flat,I}) = 0$ if and only if the map

$$(\pi^*(\gamma \circ i))_{T^\flat} : F_{T^\flat} = F \otimes_{B_{S^\flat,I}} B_{T^\flat,I} \to N_{T^\flat} = N \otimes_{B_{S^\flat,I}} B_{T^\flat,I}$$

is zero. Moreover, since $I = [1, q]$, Frobenius translates of $Y_{T^\flat,I}$ cover $Y_{T^\flat}$, hence the map above is zero if and only if

$$(\gamma \circ i)_{T^\flat} : \mathcal{F}|_{X_{T^\flat}} \to \mathcal{O}(\frac{d-1}{n-k})\Big|_{X_{T^\flat}}$$

is the zero map. Thus $Z \subseteq S$ represents $S \times_{\mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d})} H_{k,C}$, whence the claim. ☐

Therefore, we are reduced to showing such $H_{k,C}$ are not perfectoid. Let us also make the following observation.

Lemma 5.5. *There exists a Zariski closed immersion $\mathbb{P}^{k-1,\diamond} \to H_{k,C}$.*

Proof. Let $\mathscr{M}_\infty^{\mathrm{LT}}$ be the infinite-level Lubin–Tate space over C. Then $\mathbb{P}^{k-1,\diamond}$ is identified with $\mathscr{M}_\infty^{\mathrm{LT},\diamond}/\underline{\mathrm{GL}_k(K)}$. For $S \in \mathrm{Perf}_C$, this quotient parametrizes injections $\mathcal{F} \hookrightarrow \mathcal{O}(1/k)$ such that $\mathcal{F}$ is isomorphic to $\mathcal{O}^k$ at all geometric points and the cokernel is isomorphic to $\iota_* W$ for some rank 1 projective $\mathcal{O}_S$-module W [SW13]. Since the canonical divisor $\iota : S \to X_{S^\flat}$ gives a section $\infty : \mathrm{Spd}\, C \to \mathrm{Div}^1_C$ and the Plücker morphism gives a map $H_{k,C} \to \mathrm{Div}^1_C$, we see that the fibre product

$$\begin{array}{ccc} H_{k,C} \times_{\mathrm{Div}^1_C, \infty} \mathrm{Spd}\, C & \longrightarrow & H_{k,C} \\ \downarrow & & \downarrow \\ \mathrm{Spd}\, C & \xrightarrow{\quad\infty\quad} & \mathrm{Div}^1_C \end{array}$$

is exactly $\mathbb{P}^{k-1,\diamond}$. As ∞ is a Zariski closed immersion, so is the map $\mathbb{P}^{k-1,\diamond} \to H_{k,C}$. ☐

Proposition 5.6. *The Grassmannian $\mathrm{Gr}_{k,C}(\mathbf{B}^{\varphi^n=\pi^d})$ is not perfectoid if $kd \geq n$.*

Proof. By Lemma 5.4, it suffices to show $H_{k,C}$ is not perfectoid. When $k = 1$, $H_{k,C}$ is the same as Div^1_C and Proposition 5.1 gives the result. When $k \geq 2$, since $\mathbb{P}^{k-1,\diamond}$ is not perfectoid, Lemma 5.5 implies that $H_{k,C}$ is not perfectoid. ☐

6. Mod p Jacquet–Langlands correspondence

In this section, we apply Theorem 4.14 to obtain vanishing results of Scholze's mod p Jacquet–Langlands functor. Specifically, Theorem 4.14 proves the conjecture of Hansen–Mann, from which the applications follow naturally follow using tools developed in [HM22].

Recall from section 1.3, for $0 < k < n$ and $S \in \mathrm{Perf}_C$, the moduli space $W_{n,k}$ parametrizes surjections $\mathcal{O}(1/n) \twoheadrightarrow \mathcal{E}$ over $X_{S^\flat}$ with $\mathcal{E}$ isomorphic to $\mathcal{O}(1/k)$ at every geometric point. Specifically, it is isomorphic to the Grassmannian $\mathrm{Gr}_{n-k,C}(\mathbf{B}^{\varphi^n=\pi})$, see (1.5). We then deduce:

Proposition 6.1 (Conjecture 1.6)**.** *The space $W_{n,k}$ is perfectoid for all $0 < k < n$.*

Proof. This follows directly from the $d = 1$ case of Theorem 4.14 and the identification above. ☐

6.1. **Scholze's functor.** We recall Scholze's functor for the mod p Jacquet–Langlands correspondence and also Hansen–Mann's partial Künneth formula.

From the infinite-level Lubin–Tate space $\mathcal{M}_\infty^{\mathrm{LT}}$ over C, quotienting by $\mathrm{GL}_n(K)$ gives the projective space $\mathbb{P}^{n-1}$. Let $\mathrm{Rep}^{\mathrm{adm}}_{\mathbb{F}_p}(\mathrm{GL}_n(K))$ be the category of admissible smooth representations of $\mathrm{GL}_n(K)$ over $\mathbb{F}_p$. For $\rho \in \mathrm{Rep}^{\mathrm{adm}}_{\mathbb{F}_p}(\mathrm{GL}_n(K))$, Scholze [Sch18] constructed an étale sheaf $\mathcal{F}_\rho$ over $\mathbb{P}^{n-1}$ which assigns to $U \in (\mathbb{P}^{n-1})_{\text{ét}}$

$$\mathcal{F}_\rho(U) := \mathrm{Hom}_{\mathrm{cts},\mathrm{GL}_n(K)}(\left|U \times_{\mathbb{P}^{n-1}} \mathcal{M}_\infty^{\mathrm{LT}}\right|, \rho).$$

This sheaf $\mathcal{F}_\rho$ is $D^\times_{1/n}$-equivariant, and Scholze's functor is defined as

$$\mathcal{J}^i_n(\rho) := H^i_{\text{ét}}(\mathbb{P}^{n-1}, \mathcal{F}_\rho),$$

which gives an admissible $D_{1/n}^{\times}$-representation. Specifically, $\mathcal{J}_n^i$ has the following vanishing bound.

Theorem 6.2 ([Sch18, Theorem 1.1])**.** *Froe every $\rho \in \mathrm{Rep}_{\mathbb{F}_p}^{\mathrm{adm}}(\mathrm{GL}_n(K))$, $\mathcal{J}_n^i(\rho)$ vanishes for $i > 2n-2$.*

Consider the quotient stack $[\mathbb{P}^{n-1}/D_{1/n}^{\times}]$. The commuting action of $\mathrm{GL}_n(K)$ and $D_{1/n}^{\times}$ on $\mathcal{M}_{\infty}^{\mathrm{LT}}$ gives a $\mathrm{GL}_n(K)$-torsor and a $D_{1/n}^{\times}$-torsor over the stack. Write $f_n : [\mathbb{P}^{n-1}/D_{1/n}^{\times}] \to B\underline{\mathrm{GL}_n(K)}$ and $g_n : [\mathbb{P}^{n-1}/D_{1/n}^{\times}] \to B\underline{D_{1/n}^{\times}}$ for the maps coming from the corresponding torsors, where both f_n and g_n are smooth, and g_n is proper. In [HM22], Hansen-Mann considered the derived version of Scholze's functor

$$\mathcal{J}_n : D(\mathrm{GL}_n(K)) \to D(D_{1/n}^{\times}),$$

which is defined as $\mathcal{J}_n = (g_n)_{\text{ét},*}(f_n)_{\text{ét}}^*$.

For the space $W_{n,k}$, trivilizing the target bundle $\mathcal{O}(1/k)$ in its definition gives a $D_{1/k}^{\times}$-torsor over it; trivializing the bundle $\mathcal{O}^{n-k}$ from the identification (1.4) gives a $\mathrm{GL}_{n-k}(K)$-torsor. We then have

$$\begin{tikzcd} & W_{n,k} \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ B\underline{\mathrm{GL}_{n-k}(K)} & & B\underline{D_{1/k}^{\times}} \end{tikzcd}$$

For $0 < k < n$, let $P_k(K) \subset \mathrm{GL}_n(K)$ be the standard maximal parabolic subgroup of block size $(n-k, k)$. Hansen-Mann obtained the following partial Künneth formula:

Theorem 6.3 ([HM22, Theorem 5.2])**.** *For $\sigma_1 \in D(\mathrm{GL}_{n-k}(K))$ and $\sigma_2 \in D(\mathrm{GL}_k(K))$, we have*

$$\mathcal{J}_n(\mathrm{Ind}_{P_k(K)}^{\mathrm{GL}_n(K)}(\sigma_1 \boxtimes \sigma_2)) \cong R\Gamma(W_{n,k}, h_{1,\text{ét}}^*\sigma_1 \otimes h_{2,\text{ét}}^*\mathcal{J}_k(\sigma_2)).$$

6.2. **Vanishing bounds of Scholze's functor.** As one can readily see from [JLH21] and [HM22], the perfectoidness of such moduli space of morphisms between vector bundles can be used to give sharper bounds of Scholze's functor on certain mod p representations of $\mathrm{GL}_n(K)$. We now discuss the application of Proposition 6.1.

As already dictated by [HM22, Proposition 5.5], the first application of the perfectoidness of $W_{n,k}$ is the following:

Proposition 6.4. *Given $\sigma_1 \in \mathrm{Rep}_{\mathbb{F}_p}^{\mathrm{adm}}(\mathrm{GL}_{n-k}(K))$ and $\sigma_2 \in \mathrm{Rep}_{\mathbb{F}_p}^{\mathrm{adm}}(\mathrm{GL}_k(K))$, we have*

$$\mathcal{J}_n^i(\mathrm{Ind}_{P_k(K)}^{\mathrm{GL}_n(K)}(\sigma_1 \boxtimes \sigma_2)) = 0$$

for all $i > n+k-2$.

Proof. As $\mathcal{J}^i$ gives an $\mathbb{F}_p$-vector space, it is enough to prove the cohomology group in question, after tensoring with $\mathcal{O}_C/p$, is almost zero when $i > n+k-2$. We then have

$$\begin{aligned} &\mathcal{J}_n^i(\mathrm{Ind}_{P_{n-k,k}}^{\mathrm{GL}_n(K)}(\sigma_1 \boxtimes \sigma_2)) \otimes \mathcal{O}_C/p \\ &\cong^a \mathcal{J}_n^i(\mathrm{Ind}_{P_{n-k,k}}^{\mathrm{GL}_n(K)}(\sigma_1 \boxtimes \sigma_2) \otimes \mathcal{O}^+/p) \\ &\cong H^i(W_{n,k}, h_{1,\text{ét}}^*\sigma_1 \otimes h_{2,\text{ét}}^*\mathcal{J}_k(\sigma_2) \otimes \mathcal{O}^+/p) \end{aligned} \tag{6.1}$$

where the first isomorphism is given by [Sch18, Theorem 3.2] and the second one is given by [HM22, Theorem 5.2]. By Theorem 6.2, the cohomology of $\mathcal{J}_k(\sigma_2)$ vanishes in degrees larger than $2k-2$. The perfectoidness of the Grassmannian combined with [Sch92, Theorem 4.5] also implies the cohomology of the Grassmannian vanishes for degrees larger than its dimension; in this case it is $n-k$. Thus using the hypercohomology spectral sequence, (6.1) vanishes for $i > n+k-2$. $\square$

In fact, such vanishing bounds extend to all parabolically induced representations:

Theorem 6.5. *Let $0 < k < n$. For any $\rho \in \mathrm{Rep}^{\mathrm{adm}}_{\mathbb{F}_p}(P_k(K))$, we have*

$$\mathfrak{J}^i_n(\mathrm{Ind}^{\mathrm{GL}_n(K)}_{P_k(K)}(\rho)) = 0 \quad \textit{for all } i > n+k-2.$$

Proof. Projecting on to the Levi subgroup of $P_k(K)$ gives a map

$$s : B\underline{P_k(K)} \to B\underline{\mathrm{GL}_{n-k}(K)} \times B\underline{\mathrm{GL}_k(k)}.$$

Since admissible representations have trivial unipotent action, we have $\rho = s^*\overline{\rho}$ for some admissible representation $\overline{\rho}$ of $\mathrm{GL}_{n-k}(K) \times \mathrm{GL}_k(K)$. The same argument in the proof of [HM22, Theorem 5.2, Lemma 5.3] then gives a natural isomorphism

$$\mathfrak{J}_n(\mathrm{Ind}^{\mathrm{GL}_n(K)}_{P_k}(s^*\overline{\rho})) \simeq R\Gamma(W_{n,k}, (h_1, h_2)^*(\mathrm{id} \times \mathfrak{J}_k)(\overline{\rho})).$$

Now recall that $\mathfrak{J}_k = (g_k)_{\text{ét},*}(f_k)^*_{\text{ét}}$. Since $\mathrm{id} \times g_k$ is proper with $\mathbb{P}^{k-1}$-fibres, Theorem 6.2 implies $(\mathrm{id} \times \mathfrak{J}_k)(\overline{\rho})$ is concentrated in degrees $\leq 2k-2$. Using the perfectoidness of $W_{n,k}$ from Proposition 6.1 and the same argument as in Proposition 6.4, we get the claimed bounds. $\square$

Remark 6.6. In [JLH21, Theorem 5.3.1], Johansson–Ludwig showed that given a representation $\rho \in \mathrm{Rep}^{\mathrm{adm}}_{\mathbb{F}_p}(P_1(K))$,

$$\mathfrak{J}^i_n(\mathrm{Ind}^{\mathrm{GL}_n(K)}_{P_1}(\rho)) = 0 \quad \text{for all } i > n-1.$$

So the theorem above can be treated as a generalization of their result to all maximal parabolic subgroups of $\mathrm{GL}_n(K)$. Also notice that for $P_1(K)$, this bound is exactly half of Scholze's bound in Theorem 6.2. But the following example shows that one cannot expect general vanishing beyond the middle point for all $P_k(K)$. Consider $\rho = \mathrm{Ind}^{\mathrm{GL}_n(K)}_{P_k}(\mathbf{1})$. By Theorem 6.3, we have

$$\mathfrak{J}_n(\rho) \simeq R\Gamma(W_{n,k}, h^*_{2,\text{ét}}\mathfrak{J}_k(\mathbf{1})).$$

Since $\mathcal{F}_{\mathbf{1}}$ is the constant sheaf on $\mathbb{P}^{k-1}$, we get $\mathfrak{J}_k(\mathbf{1}) = R\Gamma(\mathbb{P}^{k-1}, \mathbb{F}_p)$ which has highest degree $2k-2$. Therefore, we see $\mathfrak{J}^{2k-2}_n(\rho) \neq 0$. For $k > (n+1)/2$, this shows $\mathfrak{J}^i_n$ can be nonvanishing beyond the middle degree for representations induced from $P_k(K)$.

At the moment, we do not know whether the bound given in Theorem 6.5 is sharp, and it is natural to ask the following question

Question 6.7. *For $0 < k < n$, are there representations $\rho \in \mathrm{Rep}^{\mathrm{adm}}_{\mathbb{F}_p}(P_k(K))$ such that $\mathfrak{J}^{n+k-2}_n(\mathrm{Ind}^{\mathrm{GL}_n(K)}_{P_k}(\rho)) \neq 0$?*

For any standard parabolic subgroup $P \subseteq \mathrm{GL}_n$, we let

$$\mathrm{Sp}_P = \frac{\mathrm{Ind}^{\mathrm{GL}_n(K)}_{P(K)} \mathbf{1}}{\sum_{P \subsetneq Q} \mathrm{Ind}^{\mathrm{GL}_n(K)}_{Q(K)} \mathbf{1}}$$

be its associated *generalized Steinberg representation*. Notice that when $P = \mathrm{GL}_n$, Sp_P gives the trivial representation of $\mathrm{GL}_n(K)$.

Proposition 6.8. *For each fixed $0 < k < n$ and each $i > n+k-2$, there is a $D_{1/n}^{\times}$-equivariant isomorphism*

$$\mathfrak{J}_n^i(\mathrm{Sp}_{P_k}) \cong \begin{cases} \mathbb{F}_p(-(i+1)/2) & \text{if } i \text{ is odd and } i \leq 2n-3 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since P_k is maximal, we have a short exact sequence

$$0 \to \mathbf{1} \to \mathrm{Ind}_{P_k(K)}^{\mathrm{GL}_n(K)}(\mathbf{1}) \to \mathrm{Sp}_{P_k} \to 0.$$

Then in the long exact sequence obtained from applying $\mathfrak{J}_n$, the cohomology groups for the middle term vanish in degrees $> n+k-2$ by Theorem 6.5. Thus we have

$$\mathfrak{J}_n^i(\mathrm{Sp}_{P_k}) \cong \mathfrak{J}_n^{i+1}(\mathbf{1}) = H_{\text{ét}}^{i+1}(\mathbb{P}^{n-1}, \mathbb{F}_p).$$

Since $H_{\text{ét}}^{2j}(\mathbb{P}^{n-1}, \mathbb{F}_p(j)) \cong \mathbb{F}_p$ for every $0 \leq j \leq n-1$, the claim follows. $\square$

Using the classification of irreducible admissible representations of $\mathrm{GL}_n(K)$ from [HV19] (also cf. [Her11] and [AHHV17]), we get the following vanishing result in the top degree allowed by Theorem 6.2:

Theorem 6.9. *Let $n > 1$ and let $\rho \in \mathrm{Rep}_{\mathbb{F}_p}^{\mathrm{adm}}(\mathrm{GL}_n(K))$. Suppose that ρ is irreducible, non-supersingular and infinite-dimensional. Then we have $\mathfrak{J}_n^{2n-2}(\rho) = 0$.*

Proof. For any parabolic subgroup $P \subseteq \mathrm{GL}_n$, let $P = MN$ be its Levi decomposition. For an irreducible supersingular $\sigma \in \mathrm{Rep}_{\mathbb{F}_p}^{\mathrm{adm}}(M(K))$, we let $P^\sigma = M^\sigma N^\sigma$ be the maximal parabolic subgroup of GL_n such that σ extends trivially on the unipotent subgroup. By [HV19, Theorem 4], for any irreducible $\rho \in \mathrm{Rep}_{\mathbb{F}_p}^{\mathrm{adm}}(\mathrm{GL}_n(K))$, there exists

- a parabolic subgroup $P = MN \subseteq \mathrm{GL}_n$
- a supersingular irreducible $\sigma \in \mathrm{Rep}_{\mathbb{F}_p}^{\mathrm{adm}}(M(K))$
- a parabolic subgroup $Q \subseteq \mathrm{GL}_n$ with $P \subseteq Q \subseteq P^\sigma$

such that

$$\rho \cong \mathrm{Ind}_{P^\sigma(K)}^{\mathrm{GL}_n(K)}(\mathrm{Sp}_{Q(K)}^{M^\sigma(K)}(\sigma)) \tag{6.2}$$

where

$$\mathrm{Sp}_{Q(K)}^{M^\sigma(K)}(\sigma) = \frac{\mathrm{Ind}_{Q(K)}^{M^\sigma(K)}(\sigma)}{\sum_{Q \subsetneq Q' \subseteq P^\sigma} \mathrm{Ind}_{Q'(K)}^{M^\sigma(K)}(\sigma)}$$

Since $\mathrm{Sp}_{Q(K)}^{M^\sigma(K)}(\sigma)$ admits a surjection from $\mathrm{Ind}_{Q(K)}^{M^\sigma(K)}(\sigma)$, (6.2) gives a surjection

$$\mathrm{Ind}_{Q(K)}^{\mathrm{GL}_n(K)}(\sigma) \twoheadrightarrow \rho.$$

We claim $Q \neq \mathrm{GL}_n$. If $P = \mathrm{GL}_n$, then ρ will be supersingular. If $P \neq \mathrm{GL}_n$ and $Q = \mathrm{GL}_n$, then ρ is identified with the extension of σ to $\mathrm{GL}_n(K)$. Thus $N(K)$ acts trivially on ρ. Its normal closure is $\mathrm{SL}_n(K)$, so ρ factors through $K^\times$ and gives an irreducible admissible representation of $K^\times$, which is finite-dimensional. This gives a contradiction. Then let P_k be the maximal parabolic subgroup containing Q. By Theorem 6.5, we get $\mathfrak{J}_n^{2n-2}(\mathrm{Ind}_{P_k(K)}^{\mathrm{GL}_n(K)}(\sigma)) = 0$, and right-exactness of $\mathfrak{J}_n^{2n-2}$ gives $\mathfrak{J}_n^{2n-2}(\rho) = 0$. $\square$

Remark 6.10. Since Steinberg representations are non-supersingular and infinite-dimensional, Proposition 6.8 gives examples of such representations considered in the theorem above with nonvanishing $\mathfrak{J}^{2n-3}$. In particular, the bound $2n-2$ is sharp.

Remark 6.11. As pointed out to us by Vignéras, the assumption of ρ being infinite-dimensional is crucial. From the proof above, we see that every finite-dimensional $\rho \in \mathrm{Rep}^{\mathrm{adm}}_{\mathbb{F}_p}(\mathrm{GL}_n(K))$ factors through the determinant. If ρ is irreducible, it is then obtained by restriction of scalars from an admissible character $\chi \circ \det : \mathrm{GL}_n(K) \to \mathbb{F}^\times$ for some finite extension $\mathbb{F}/\mathbb{F}_p$. For a proper parabolic subgroup $P = MN$, let $\overline{P}$ denote its opposite parabolic subgroup and let $\mathrm{Ord}_{\overline{P}}$ denote Emerton's ordinary part functor ([Eme10, §3.1]). Since the unipotent radical acts trivially on ρ, the Hecke operator defining $\mathrm{Ord}_{\overline{P}}$ is 0 in characteristic p, hence we have $\mathrm{Ord}_{\overline{P}}(\rho) = 0$. As $\mathrm{Ord}_{\overline{P}}$ is right adjoint to $\mathrm{Ind}_{P(K)}^{\mathrm{GL}_n(K)}$ ([Eme10, Theorem 4.4.6]), we get $\mathrm{Hom}_{\mathrm{GL}_n(K)}(\mathrm{Ind}_{P(K)}^{\mathrm{GL}_n(K)}(\sigma), \rho) = 0$ for any $\sigma \in \mathrm{Rep}^{\mathrm{adm}}_{\mathbb{F}_p}(P(K))$. In particular, ρ cannot be a quotient of a proper parabolic induction.

In the meantime, these finite-dimensional non-supersingular representations also give genuine exceptions to the vanishing statement. Since ρ factors through the determinant, $\mathcal{F}_\rho$ is constant on $\mathbb{P}^{n-1}$, cf. proof of [Sch18, Proposition 4.7]. Hence we have $\dim_{\mathbb{F}_p} \mathcal{J}^{2n-2}(\rho) = \dim_{\mathbb{F}_p} \rho > 0$.

References

[AHHV17] N. Abe, G. Henniart, F. Herzig, and M.-F. Vignéras, "A classification of irreducible admissible $\bmod p$ representations of p-adic reductive groups", *J. Amer. Math. Soc.* **30** no. 2 (2017), pp. 495–559 (cit. on p. 32).

[BFHHL+22] Christopher Birkbeck, Tony Feng, David Hansen, Serin Hong, Qirui Li, Anthony Wang, and Lynnelle Ye, "Extensions of vector bundles on the Fargues-Fontaine curve", *J. Inst. Math. Jussieu* **21** no. 2 (2022), pp. 487–532 (cit. on pp. 6, 28).

[Bou70] N. Bourbaki, *Éléments de mathématique. Algèbre. Chapitres 1 à 3*, Hermann, Paris, 1970 (cit. on pp. 20, 21).

[BHHMS25] Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra, and Benjamin Schraen, "Multivariable $(\varphi, \mathcal{O}_K^\times)$-modules and local-global compatibility", *Math. Ann.* **392** no. 2 (2025), pp. 2709–2801 (cit. on p. 10).

[CGJ23] Ana Caraiani, Daniel R. Gulotta, and Christian Johansson, "Vanishing theorems for Shimura varieties at unipotent level", *J. Eur. Math. Soc. (JEMS)* **25** no. 3 (2023), pp. 869–911 (cit. on p. 11).

[Col02] Pierre Colmez, "Espaces de Banach de dimension finie", *J. Inst. Math. Jussieu* **1** no. 3 (2002), pp. 331–439 (cit. on p. 1).

[Eme10] Matthew Emerton, "Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties", *Astérisque* no. 331 (2010), pp. 355–402 (cit. on p. 33).

[Far20] Laurent Fargues, "Simple connexité des fibres d'une application d'Abel-Jacobi et corps de classes local", *Ann. Sci. Éc. Norm. Supér. (4)* **53** no. 1 (2020), pp. 89–124 (cit. on pp. 1, 4–6, 27).

[FF18] Laurent Fargues and Jean-Marc Fontaine, "Courbes et fibrés vectoriels en théorie de Hodge p-adique", *Astérisque* no. 406 (2018), With a preface by Pierre Colmez, pp. xiii+382 (cit. on pp. 1, 5, 9, 10).

[FS24] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, 2024, arXiv: `2102.13459 [math.RT]`, URL: `https://arxiv.org/abs/2102.13459` (cit. on pp. 1, 4, 5, 25, 27, 28).

[Han16] David Hansen, *Quotients of adic spaces by finite groups*, Math. Res. Lett., to appear, 2016, preprint (cit. on pp. 11, 16).

[HK26] David Hansen and Kiran Kedlaya, *Sheafiness criteria for Huber rings*, 2026, URL: `https://kskedlaya.org/papers/criteria.pdf`, preprint (cit. on p. 27).

[HM22] David Hansen and Lucas Mann, *p-adic sheaves on classifying stacks, and the p-adic Jacquet-Langlands correspondence*, 2022, arXiv: 2207.04073 [math.NT], URL: https://arxiv.org/abs/2207.04073 (cit. on pp. 1–4, 7, 8, 29–31).

[HV19] G. Henniart and M.-F. Vignéras, "Representations of a p-adic group in characteristic p", *Representations of reductive groups*, vol. 101, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 2019, pp. 171–210 (cit. on pp. 3, 32).

[Her11] Florian Herzig, "The classification of irreducible admissible mod p representations of a p-adic GL_n", *Invent. Math.* **186** no. 2 (2011), pp. 373–434 (cit. on p. 32).

[JLH21] Christian Johansson, Judith Ludwig, and David Hansen, "A quotient of the Lubin-Tate tower II", *Math. Ann.* **380** no. 1-2 (2021), pp. 43–89 (cit. on pp. 4, 6–8, 24, 30, 31).

[Ked19] Kiran S. Kedlaya, "Sheaves, shtukas, and stacks", *Perfectoid Spaces: Lectures from the 2017 Arizona Winter School*, 2019 (cit. on p. 27).

[KL15] Kiran S. Kedlaya and Ruochuan Liu, "Relative p-adic Hodge theory: foundations", *Astérisque* no. 371 (2015), p. 239 (cit. on p. 23).

[LB18] Arthur-César Le Bras, "Espaces de Banach-Colmez et faisceaux cohérents sur la courbe de Fargues-Fontaine", *Duke Math. J.* **167** no. 18 (2018), pp. 3455–3532 (cit. on p. 1).

[Sch92] Claus Scheiderer, "Quasi-augmented simplicial spaces, with an application to cohomological dimension", *J. Pure Appl. Algebra* **81** no. 3 (1992), pp. 293–311 (cit. on p. 31).

[Sch18] Peter Scholze, "On the p-adic cohomology of the Lubin-Tate tower", *Ann. Sci. Éc. Norm. Supér. (4)* **51** no. 4 (2018), With an appendix by Michael Rapoport, pp. 811–863 (cit. on pp. 3, 29, 30, 33).

[Sch26] Peter Scholze, *Etale cohomology of diamonds*, 2026, arXiv: 1709.07343 [math.AG], URL: https://arxiv.org/abs/1709.07343 (cit. on pp. 6, 16, 18, 19).

[SW13] Peter Scholze and Jared Weinstein, "Moduli of p-divisible groups", *Camb. J. Math.* **1** no. 2 (2013), pp. 145–237 (cit. on pp. 2, 5, 7, 29).

[SW20] Peter Scholze and Jared Weinstein, *Berkeley lectures on p-adic geometry*, vol. 207, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 2020, pp. x+250 (cit. on pp. 26–28).

[Wei13] Jared Weinstein, *Arizona Winter School Lecture Notes: Modular curves at infinite level*, 2013, URL: https://math.bu.edu/people/jsweinst/AWS/AWSLectureNotes.pdf (cit. on p. 13).

[Wei17] Jared Weinstein, "$\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ as a geometric fundamental group", *Int. Math. Res. Not. IMRN* no. 10 (2017), pp. 2964–2997 (cit. on pp. 8, 27).

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